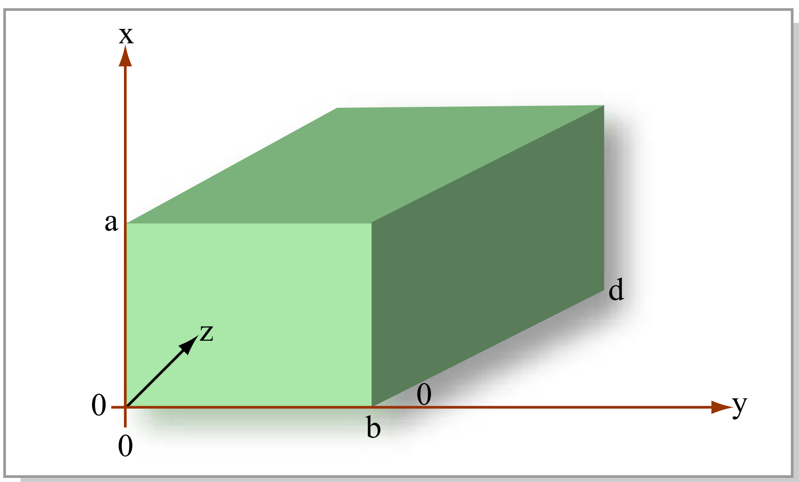
6.013 - Electromagnetics and Applications	Fall 2005
Lecture 23 - Acoustic Resonators	
Prof. Markus Zahn	December 13, 2005

I. Governing Acoustic Equations

$$\rho_0 \frac{\partial \bar{v}}{\partial t} = -\nabla p$$
$$\frac{\partial p}{\partial t} = -\rho_0 c_s^2 \nabla \cdot \bar{v}$$
$$\Rightarrow \nabla^2 p = \frac{1}{c_s^2}\frac{\partial^2 p}{\partial t^2}$$

II. Acoustic Resonator - Closed Box



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A. Boundary Conditions

$$v_x(x=0) = v_x(x=a) = 0$$
$$v_y(y=0) = v_y(y=b) = 0$$
$$v_z(z=0) = v_z(z=d) = 0$$

B. Solutions to Acoustic Equations

$$\nabla^2 p = \frac{1}{c_s^2}\frac{\partial^2 p}{\partial t^2}$$

$$p(x,y,z,t) = \text{Re}\left[\hat{p}\, e^{j(\omega t - \bar{k}\cdot\bar{r})}\right] = \text{Re}\left[\hat{p}\, e^{j(\omega t - k_x x - k_y y - k_z z)}\right]$$

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{c_s^2}\frac{\partial^2 p}{\partial t^2}$$

$$\Rightarrow -\hat{p}\left[k_x^2 + k_y^2 + k_z^2\right] = -\frac{\hat{p}\omega^2}{c_s^2}$$
$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c_s^2}$$

$$\rho_0 \frac{\partial \bar{v}}{\partial t} = -\nabla \rho$$
$$\rho_0 \frac{\partial v_x}{\partial t} = -\frac{\partial p}{\partial x}$$
$$\rho_0 \frac{\partial v_y}{\partial t} = -\frac{\partial p}{\partial y}$$
$$\rho_0 \frac{\partial v_z}{\partial t} = -\frac{\partial p}{\partial z}$$

$$v_x \propto \sin(k_x x), v_x(x=0) = 0 \Rightarrow p \propto \cos(k_x x)$$
$$v_y \propto \sin(k_y y), v_y(y=0) = 0 \Rightarrow p \propto \cos(k_y y)$$
$$v_z \propto \sin(k_z z), v_z(z=0) = 0 \Rightarrow p \propto \cos(k_z z)$$

$$p(x,y,z,t) = \mathrm{Re}\left[\hat{p}\cos(k_x x)\cos(k_y y)\cos(k_z z)e^{j\omega t}\right]$$
$$\rho_0 j\omega \hat{v}_x = \hat{p}k_x \sin(k_x x)\cos(k_y y)\cos(k_z z)$$
$$\Rightarrow \hat{v}_x = \frac{\hat{p}k_x}{\rho_0 j\omega}\sin(k_x x)\cos(k_y y)\cos(k_z z)$$
$$\hat{v}_y = \frac{\hat{p}k_y}{\rho_0 j\omega}\cos(k_x x)\sin(k_y y)\cos(k_z z)$$
$$\hat{v}_z = \frac{\hat{p}k_z}{\rho_0 j\omega}\cos(k_x x)\cos(k_y y)\sin(k_z z)$$

$$\hat{v}_x(x=a) = 0 \Rightarrow k_x a = m\pi \Rightarrow k_x = \frac{m\pi}{a}$$
$$\hat{v}_y(y=b) = 0 \Rightarrow k_y b = n\pi \Rightarrow k_y = \frac{n\pi}{b}$$
$$\hat{v}_z(z=d) = 0 \Rightarrow k_z d = p\pi \Rightarrow k_z = \frac{p\pi}{d}$$

$$\pi\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2\right]^{1/2} = \frac{\omega}{c_s}$$

$$f_{mnp} = \frac{c_s}{2}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2\right]^{1/2}$$
$$c_s \approx 330 \text{ m/s}, a = .203 \text{ m}\,(8 \text{ in}), b = .203 \text{ m}\,(8 \text{ in}), d = .305\text{m}\,(12 \text{ in})$$

(The value for speed of sound and the determined frequencies are for waves in air.)

m	n	p	f_{mnp}(Hz)	f_{measured}(Hz)
0	0	1	541	550
0	1	0	813	
0	0	2	1082	
1	1	0	1150	1150
1	1	1	1270	
1	1	2	1579	
0	0	3	1623	
0	2	0	1626	
0	1	3	1815	
1	2	2	2115	

C. Comparison of Theory to Experiment

Air -

$$\text{T} = 0^\circ\ \text{C}, p_0 = 1\ \text{atm} = 1.01 \cdot 10^5\ \text{Nt/m}^2 \qquad (1\ \text{Nt/m}^2 = 1\ \text{Pascal})$$
$$\rho_0 = 1.293\ \text{g/L} = 1.293\ \text{kg/m}^3$$
$$c_s = \sqrt{\frac{\gamma p_0}{\rho_0}} = 331\ \text{m/s}$$

Helium -

$$\text{T} = 0^\circ\ \text{C}, p_0 = 1\ \text{atm} = 1.01 \cdot 10^5\ \text{Pascals}, \gamma = 1.67$$
$$\rho_0 = 0.178\ \text{g/L} = 0.178\ \text{kg/m}^3$$
$$c_s = \sqrt{\frac{\gamma p_0}{\rho_0}} = 973\ \text{m/s}$$

Helium/air mixture -

$$f_{001} = \frac{c_s}{2d} = 950\ \text{Hz was measured}$$
$$c_s = 2d(950) = 2(.305)950 = 580\ \text{m/s}$$
$$c_{s,\text{air}} < c_{s,\text{air/He mixture}} < c_{s,\text{He}}$$
$$331 < 580 < 973\text{m/s}$$

III. Acoustic Resonator - $z = 0$ side open, other 5 sides closed

A. Boundary Conditions

$$v_x(x = 0) = v_x(x = a) = 0$$
$$v_y(y = 0) = v_y(y = b) = 0$$
$$v_z(z = d) = 0,\ p(z = 0) = 0$$

B. Solutions

$$p(x, y, z, t) = \text{Re}\left[\hat{p}\cos(k_x x)\cos(k_y y)\sin(k_z z)e^{j\omega t}\right]$$
$$\hat{v}_x = \frac{\hat{p}k_x}{\rho_0 j\omega}\sin(k_x x)\cos(k_y y)\sin(k_z z)$$
$$\hat{v}_y = \frac{\hat{p}k_y}{\rho_0 j\omega}\cos(k_x x)\sin(k_y y)\sin(k_z z)$$
$$\hat{v}_z = -\frac{\hat{p}k_z}{\rho_0 j\omega}\cos(k_x x)\cos(k_y y)\cos(k_z z)e^{j\omega t}$$

$$
\begin{aligned}
\hat{v}_x(x=0) &= 0 \\
\hat{v}_x(x=a) &= 0 \Rightarrow k_x a = m\pi \Rightarrow k_x = \frac{m\pi}{a} \\
\hat{v}_y(y=0) &= 0 \\
\hat{v}_y(y=b) &= 0 \Rightarrow k_y b = n\pi \Rightarrow k_y = \frac{n\pi}{b} \\
\hat{v}_z(z=d) &= 0 \Rightarrow k_z d = (2p+1)\frac{\pi}{2} \Rightarrow k_z = \frac{(2p+1)\pi}{2d} \\
\hat{p}(z=0) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\omega^2}{c_s^2} &= \pi^2 \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{2p+1}{2d}\right)^2 \right] \\
f_{mnp} &= \frac{c_s}{2} \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{2p+1}{2d}\right)^2 \right]^{1/2} \\
f_{000} &= \left.\frac{(2p+1)c_s}{4d}\right|_{p=0}, c_s = 330\text{m/s (air)}, d = .305\ \text{m} \Rightarrow f_{000} = 270.5\ \text{Hz} \\
f_{100} &= \frac{c_s}{2} \left[\left(\frac{1}{a}\right)^2 + \left(\frac{1}{2d}\right)^2 \right]^{1/2} = 856.6\ \text{Hz}
\end{aligned}
$$