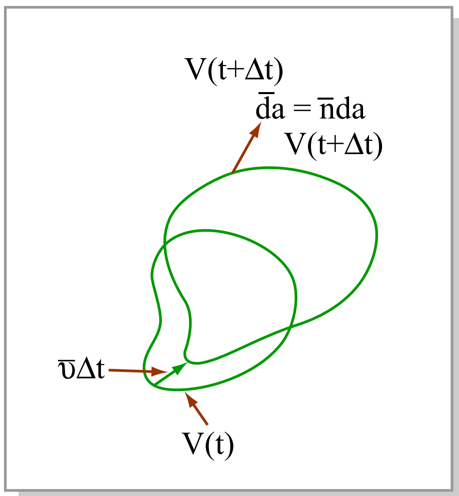
6.013 - Electromagnetics and Applications	Fall 2005

Lecture 22 - Acoustic Waves

Prof. Markus Zahn December 8, 2005

I. Useful identity (ζ - any quantity)

$$\frac{d}{dt}\int_V dV\,\zeta = \frac{1}{\Delta t}\left[\int_{V(t+\Delta t)} dV\,\zeta(t+\Delta t) - \int_{V(t)} dV\,\zeta(t)\right]$$



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$$\begin{aligned}
\frac{d}{dt}\int_V dV\,\zeta &= \frac{1}{\Delta t}\int_{V(t)} dV\,\left[\zeta(t+\Delta t)-\zeta(t)\right] + \frac{1}{\Delta t}\int_{\Delta V=\bar{d}a\cdot\bar{v}\Delta t} dV\,\zeta(t+\Delta t)\\
&= \int_{V(t)} dV\,\frac{\partial\zeta}{\partial t} + \oint_S \bar{d}a\cdot\bar{v}\,\zeta(t+\Delta t)\\
&= \int_{V(t)} dV\,\frac{\partial\zeta}{\partial t} + \int_{V(t)} dV\,\nabla\cdot\left[\zeta(t)\bar{v}\right]\\
&= \int_{V(t)} dV\,\left[\frac{\partial\zeta}{\partial t} + \nabla\cdot(\zeta\bar{v})\right]
\end{aligned}$$

II. Conservation of Mass (ρ - mass density)

$$\frac{d}{dt}\int_V dV\,\rho = 0 = \int_V dV\,\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\bar{v})\right]$$

Since V is arbitrary: $\dfrac{\partial\rho}{\partial t} + \nabla\cdot(\rho\bar{v}) = 0$ (Conservation of mass)

III. Conservation of Momentum, ith component ($i = x, y$, or z)

$$\frac{d}{dt}\int_V dV\,\rho v_i = \int_V dV\left[\frac{\partial}{\partial t}(\rho v_i) + \nabla\cdot(\rho v_i \bar{v})\right] = \int_V dV\,\underbrace{F_{Ti}}_{\substack{\text{Total force}\\ \text{density}}}$$

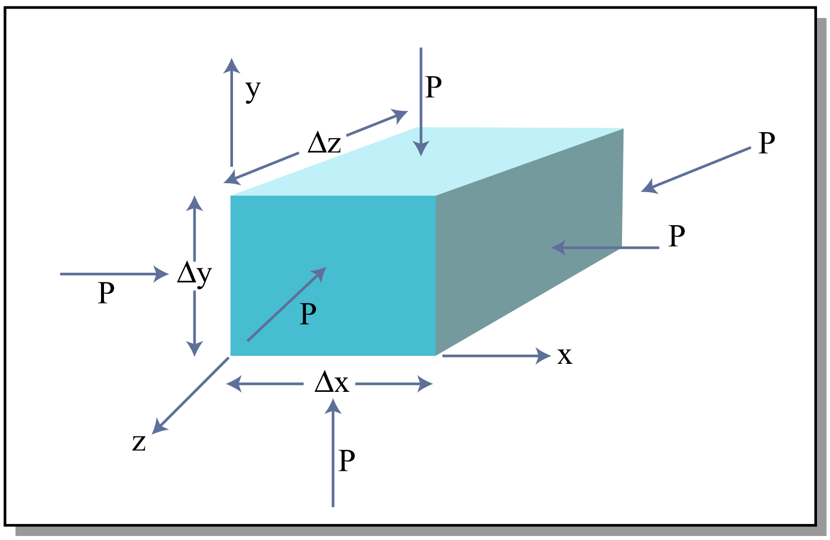
$$\frac{\partial}{\partial t}(\rho v_i) + \nabla\cdot(\rho v_i\bar{v}) = F_{Ti}$$

$$v_i\frac{\partial \rho}{\partial t} + \rho\frac{\partial v_i}{\partial t} + v_i\nabla\cdot(\rho\bar{v}) + \rho(\bar{v}\cdot\nabla)v_i = F_{Ti}$$

$$v_i\Big[\underbrace{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\bar{v})}_{\substack{=0\\ \text{(mass conservation)}}}\Big] + \rho\left(\frac{\partial v_i}{\partial t} + (\bar{v}\cdot\nabla)v_i\right) = F_{Ti}$$

$$\rho\left[\frac{\partial\bar{v}}{\partial t} + (\bar{v}\cdot\nabla)\bar{v}\right] = \bar{F}_T$$

IV. Force density due to pressure (force/area)



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$$\bar{F}_p = \{-[p(x+\Delta x) - p(x)]\,\bar{i}_x\Delta y\Delta z - [p(y+\Delta y) - p(y)]\,\bar{i}_y\Delta x\Delta z - [p(z) - p(z-\Delta z)]\,\bar{i}_z\Delta x\Delta y\}\,\frac{1}{\Delta x\Delta y\Delta z}$$

$$= -\frac{[p(x+\Delta x) - p(x)]}{\Delta x}\bar{i}_x - \frac{[p(y+\Delta y) - p(y)]}{\Delta y}\bar{i}_y - \frac{[p(z) - p(z-\Delta z)]}{\Delta z}\bar{i}_z$$

$$= -\left[\frac{\partial p}{\partial x}\bar{i}_x + \frac{\partial p}{\partial y}\bar{i}_y + \frac{\partial p}{\partial z}\bar{i}_z\right]$$

$$= -\nabla p$$

V. Governing Fluid Equations

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\bar{v}) = 0$$

$$\rho\left[\frac{\partial\bar{v}}{\partial t} + (\bar{v}\cdot\nabla)\bar{v}\right] = -\nabla p$$

VI. Small Perturbations About Equilibrium of Stationary Fluid

$$\begin{aligned} \rho &= \rho_0 + \rho' \qquad (\rho' \ll \rho_0) \\ \bar{v} &= 0 + \bar{v}' \\ p &= p_0 + p' \end{aligned}$$

$$\begin{aligned} &\tfrac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \bar{v}' = 0 \\ \rho_0 &\left[\tfrac{\partial \bar{v}'}{\partial t} + \underbrace{(\bar{v}' \cdot \nabla)\bar{v}'}_{\substack{=0 \\ \text{second order}}}\right] = -\nabla p' \end{aligned} \quad \Rightarrow \quad \begin{aligned} &\tfrac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \bar{v}' = 0 \\ &\rho_0 \tfrac{\partial \bar{v}'}{\partial t} = -\nabla p' \end{aligned}$$

VII. Pressure / Density Constitutive Law

A. Ideal Gas - $p = \rho R T$, R is the Gas Constant $= R_g /$ molecular weight in grams

1. Isothermal (T constant) $R_g = 8.31 \times 10^3 \frac{\text{Joules}}{\text{kg (mole) K}}$

$$\begin{aligned} p &= \rho R T \\ p' &= R T \rho' \end{aligned}$$

where "mole" indicates the molecular weight in grams.

2. Adiabatic

$$\frac{\partial p}{\partial \rho} = \frac{\gamma p}{\rho} \Rightarrow p = \text{constant } \rho^\gamma \Rightarrow p' = \frac{\gamma p_0}{\rho_0}\rho'$$

$$\gamma = \frac{c_p}{c_v} = \text{ ratio of specific heats } = \frac{5}{3}(\text{monatomic ideal gas})$$

B. Liquid or Solid

$$\frac{\partial p}{\partial \rho} = \frac{\kappa}{\rho} \Rightarrow p' = \frac{\kappa}{\rho_0}\rho', \text{ where } \kappa \text{ is the Bulk Modulus}$$

VIII. Acoustic Wave Equation

$$\nabla \cdot \left\{\rho_0 \frac{\partial \bar{v}}{\partial t} = -\nabla p'\right\} \Rightarrow \rho_0 \frac{\partial}{\partial t}(\nabla \cdot \bar{v}') = -\nabla \cdot (\nabla p') = -\nabla^2 p'$$

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \bar{v}' = 0 \Rightarrow \nabla \cdot \bar{v}' = -\frac{1}{\rho_0}\frac{\partial \rho'}{\partial t}$$

$$+\nabla^2 p' = +\frac{\partial^2 \rho'}{\partial t^2}$$

$$c_s = \left[\frac{p'}{\rho'}\right]^{1/2} \qquad \left(\text{Units: } \frac{\text{nt}}{\text{m}^2 \frac{\text{kg}}{\text{m}^3}} = \frac{\text{nt-m}}{\text{kg}} = \frac{\text{m}^2}{\text{s}^2} = (\text{velocity})^2\right.$$

$$c_s = \begin{cases} \sqrt{RT} & \text{Isothermal Ideal Gas} \\ \sqrt{\frac{\gamma p_0}{\rho_0}} & \text{Adiabatic Ideal Gas} \\ \sqrt{\frac{\kappa}{\rho_0}} & \text{Liquid or Solid} \end{cases}$$

In air: (Adiabatic, $\gamma = 1.4$), $\rho_0 = 1.29\ \text{kg/m}^3$, $p_0 = 1.01 \times 10^5 \frac{\text{nt}}{\text{m}^2}$ (1 atmosphere), $c_s = \sqrt{\frac{\gamma p_0}{\rho_0}} \approx$ 330 m/s

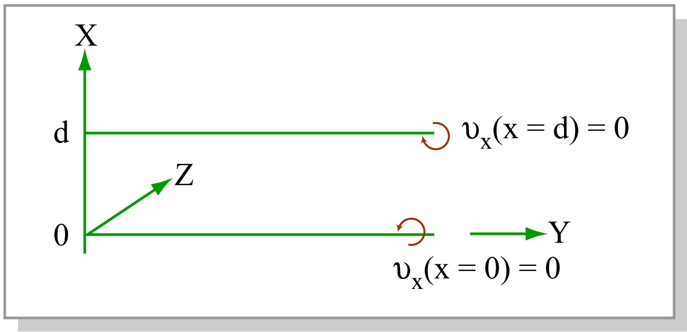
In water: $c_s \approx 1500$ m/s

$$\rho'^2 = \frac{p'^2}{c_s^2} \Rightarrow \nabla^2 p' = \frac{1}{c_s^2}\frac{\partial^2 p'}{\partial t^2}\ (c_s \text{ is the speed of sound})$$

$$\nabla^2 p' = \frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} + \frac{\partial^2 p'}{\partial z^2}$$

IX. Acoustic Waveguide

A. Parallel plate waveguide



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$$p' = \text{Re}\left[\hat{p}(x)e^{j(\omega t - k_z z)}\right]$$

$$\nabla^2 p' = \frac{1}{c_s^2}\frac{\partial^2 p'}{\partial t^2} \Rightarrow \frac{d^2\hat{p}}{dx^2} - k_z^2\hat{p} = -\frac{\omega^2}{c_s^2}\hat{p}$$

$$\frac{d^2\hat{p}}{dx^2} + \underbrace{\left(\frac{\omega^2}{c_s^2} - k_z^2\right)}_{k_x^2}\hat{p} = 0$$

$$\frac{d^2\hat{p}}{dx^2} + k_x^2\hat{p} = 0 \Rightarrow \hat{p}(x) = A\sin(k_x x) + B\cos(k_x x)$$

$$\rho_0\frac{\partial \bar{v}'}{\partial t} = -\nabla p' \Rightarrow \rho_0 j\omega\hat{v}_x = -\frac{d\hat{p}}{dx} = -k_x\left[A\cos(k_x x) - B\sin(k_x x)\right]$$

$$\rho_0 j\omega\hat{v}_z = jk_z\hat{p}$$

$$\hat{v}_x(x=0) = 0 \Rightarrow A = 0$$

$$\hat{v}_x = \frac{Bk_x}{\rho_0 j\omega}\sin(k_x x)$$

$$\hat{v}_x(x=d) = 0 \Rightarrow \sin(k_x d) = 0 \Rightarrow k_x d = m\pi, m = 0, 1, 2, \ldots$$

$$\hat{p}(x) = B\cos(k_x x)$$

$$\bar{v} = v_x\bar{i}_x + v_z\bar{i}_z = \frac{Bk_x}{\rho_0 j\omega}\sin(k_x x)\bar{i}_x + \frac{Bk_z}{\rho_0\omega}\cos(k_x x)\bar{i}_z$$

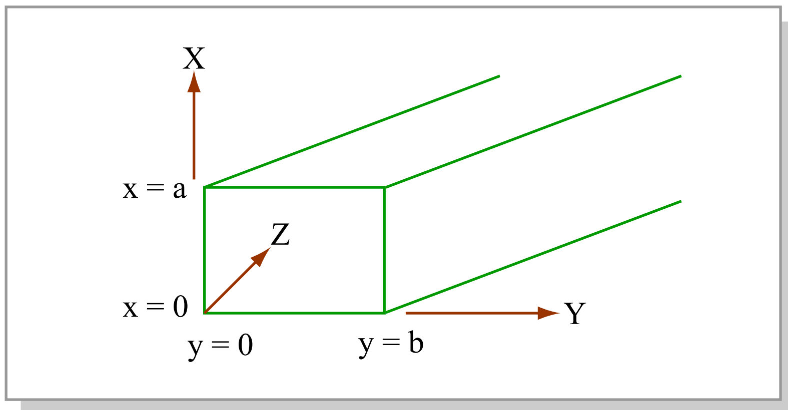
$$k_x^2 + k_z^2 = \frac{\omega^2}{c_s^2} = \left(\frac{m\pi}{d}\right)^2 + k_z^2$$

$$k_z = \sqrt{\frac{\omega^2}{c_s^2} - \left(\frac{m\pi}{d}\right)^2}$$

$m = 0$ - TEM mode

$$\bar{v} = \frac{Bk_z}{\rho_0\omega}\bar{i}_z = \frac{B}{\rho_0 c_s}\bar{i}_z$$
$$p = B$$
$$\eta_s = \frac{p}{v_z} = \rho_0 c_s \text{ is the Acoustic Impedance}$$

B. Rectangular Acoustic Waveguide



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B.C.

$$u_x(x=0) = u_x(x=a) = 0$$
$$u_y(y=0) = u_y(y=b) = 0$$

$$p' = \mathrm{Re}\left[\hat{p}(x,y)e^{j(\omega t - k_z z)}\right]$$
$$\hat{p}(x,y) = [A\sin(k_x x) + B\cos(k_x x)]\,[C\sin(k_y y) + D\cos(k_y y)]\ \left(k_x^2 + k_y^2 + k_z^2 = \omega^2/c_s^2\right)$$
$$\hat{v}_x = -\frac{1}{\rho_0 j\omega}\frac{\partial\hat{p}}{\partial x}$$
$$\hat{v}_y = -\frac{1}{\rho_0 j\omega}\frac{\partial\hat{p}}{\partial y}$$
$$\hat{v}_z = \frac{k_z}{\rho_0\omega}\hat{p}$$
$$\hat{p} = A\cos(k_x x)\cos(k_y y)$$
$$\hat{\bar{v}} = -\frac{A}{\rho_0 j\omega}\left[-\bar{i}_x k_x \sin(k_x x)\cos(k_y y) - \bar{i}_y k_y \cos(k_x x)\sin(k_y y) + k_z\bar{i}_z\cos(k_x x)\cos(k_y y)\right]$$
$$k_x = \frac{m\pi}{a}, m = 0, 1, 2, \ldots$$
$$k_y = \frac{n\pi}{b}, n = 0, 1, 2, \ldots$$
$$k_z = \sqrt{\frac{\omega^2}{c_s^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

X. Poynting Theorem

$$\bar{v}' \cdot \left(\rho_0 \frac{\partial \bar{v}'}{\partial t} = -\nabla p'\right) \Rightarrow \frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 |\bar{v}'|^2\right) = -(\bar{v}' \cdot \nabla) p'$$

$$p' \left(\nabla \cdot \bar{v}' = -\frac{1}{\rho_0}\frac{\partial \rho'}{\partial t} = -\frac{1}{\rho_0 c_s^2}\frac{\partial p'}{\partial t}\right) \Rightarrow p'(\nabla \cdot \bar{v}') = -\frac{\partial}{\partial t}\left(\frac{1}{2\rho_0 c_s^2} p'^2\right)$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 |\bar{v}'|^2 + \frac{1}{2}\frac{p'^2}{\rho_0 c_s^2}\right) = -\nabla \cdot (p' \bar{v}')$$

$$\text{Integral form: } \oint_S \bar{da} \cdot p' \bar{v}' = -\frac{d}{dt}\int_V dV \left[\frac{1}{2}\rho_0 |\bar{v}'|^2 + \frac{1}{2}\frac{p'^2}{\rho_0 c_s^2}\right]$$