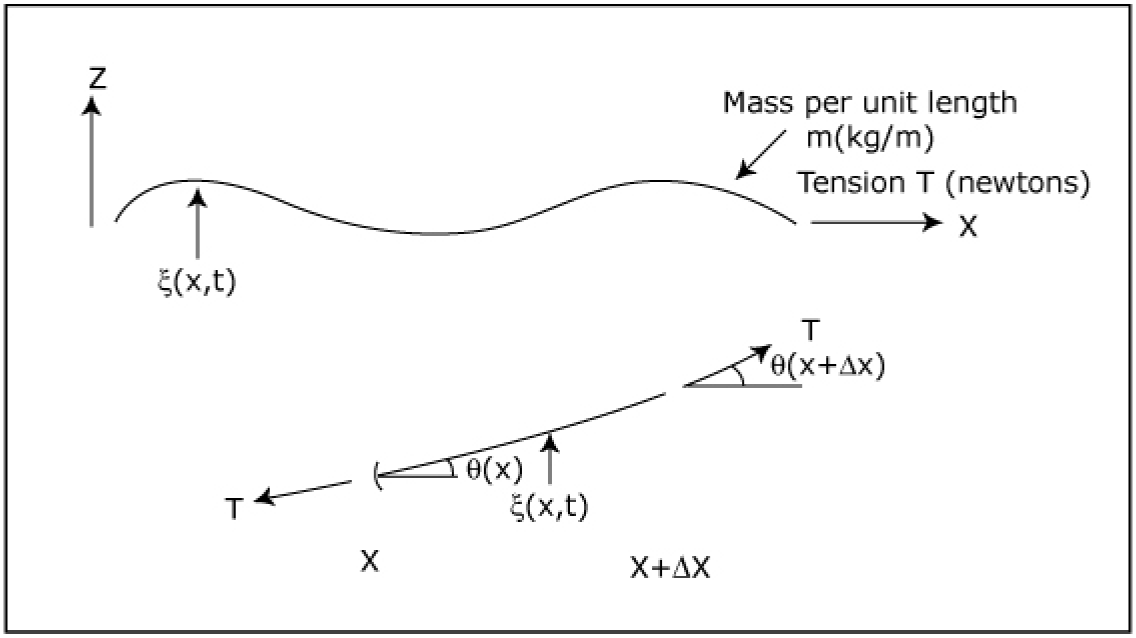
6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

# Lecture 18: Waves and Instabilities In Elastic Media

## I. Transverse Motions of Wires Under Tension




Force per unit length (nt/m)

$$m\cancel{\Delta x}\,\frac{\partial^2 \xi}{\partial t^2} = \frac{T_z(x+\Delta x) - T_z(x)}{\Delta x} + F_{ext}\,\cancel{\Delta x}$$

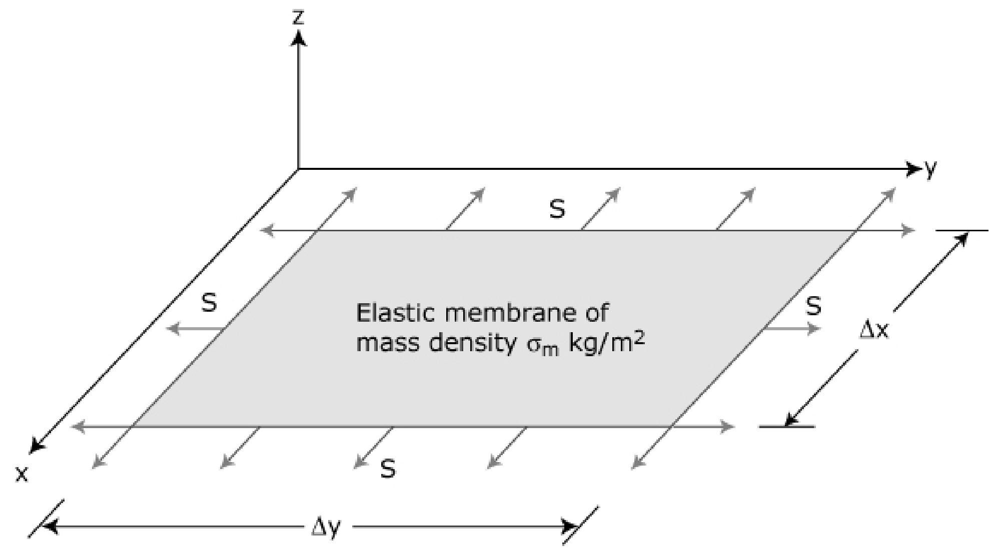
$$m\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial T_z}{\partial x} + F_{ext}$$

$$T_z = T\sin\theta \approx T\tan\theta \approx T\left[\frac{\xi(x+\Delta x) - \xi(x)}{\Delta x}\right]$$

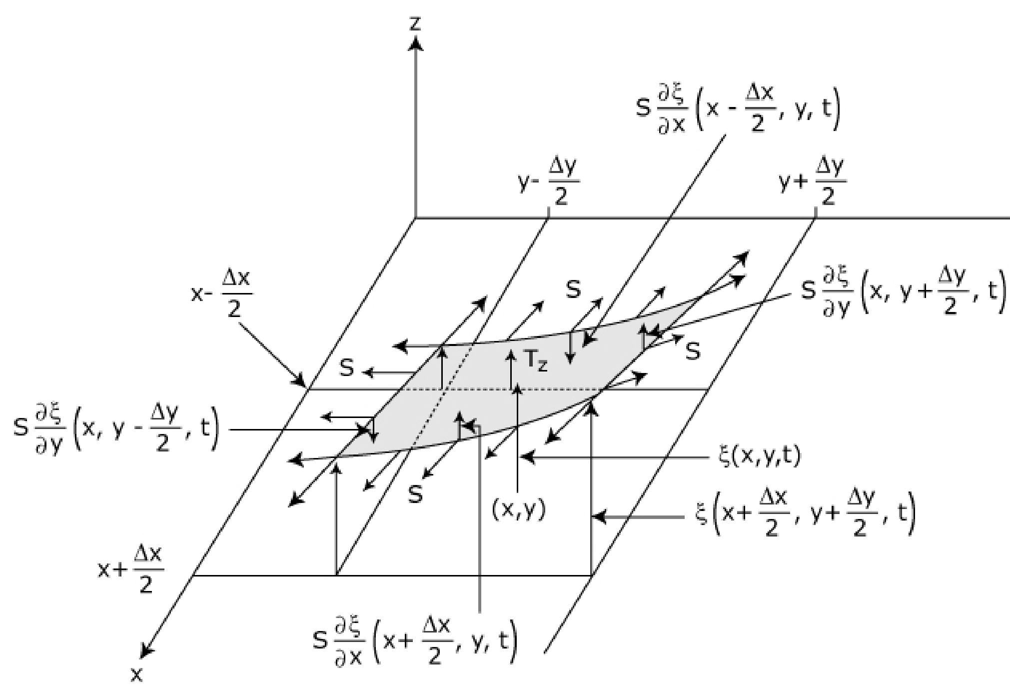
$$\approx T\frac{\partial \xi}{\partial x}$$

$$m\frac{\partial^2 \xi}{\partial t^2} = T\frac{\partial^2 \xi}{\partial x^2} + F_{ext}$$

## II. Transverse Motions of Membranes




A plane-elastic membrane in equilibrium subject to a tension S N/m along its edges.




Section of membrane having area ($\Delta x\ \Delta y$) and subject to the uniform tension S. The displacement at the center of the section (x,y) is $\xi(x,y,t)$.

$$\sigma_m \,\Delta x\, \Delta y\, \frac{\partial^2 \xi}{\partial t^2} = S\,\Delta x\left[\frac{\partial \xi}{\partial y}\left(x,\, y+\frac{\Delta y}{2},\, t\right) - \frac{\partial \xi}{\partial y}\left(x,\, y-\frac{\Delta y}{2},\, t\right)\right]$$

↑ $\sigma_m$: membrane mass per unit area $\left(kg/m^2\right)$

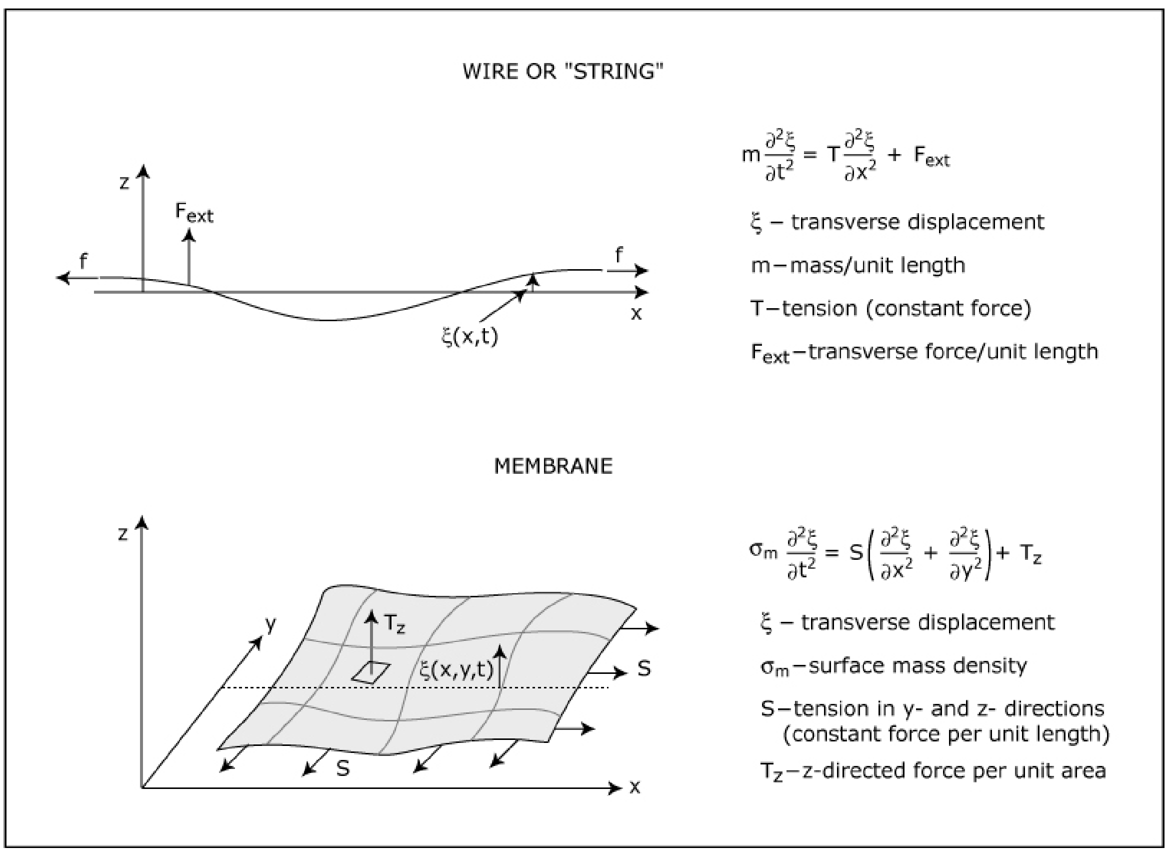
$$+S\,\Delta y\left[\frac{\partial \xi}{\partial x}\left(x+\frac{\Delta x}{2}, y, t\right) - \frac{\partial \xi}{\partial x}\left(x-\frac{\Delta x}{2}, y, t\right)\right]$$

↑ $S$: membrane tension(newtons/m)

$$+T_z\,\Delta x\,\Delta y$$

↑ $T_z$: external force per unit area (newtons/$m^2$)

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S\left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2}\right) + T_z$$



## III. Non-Dispersive Waves on a String $\left(F_{ext} = 0\right)$

### A. Dispersion Equation

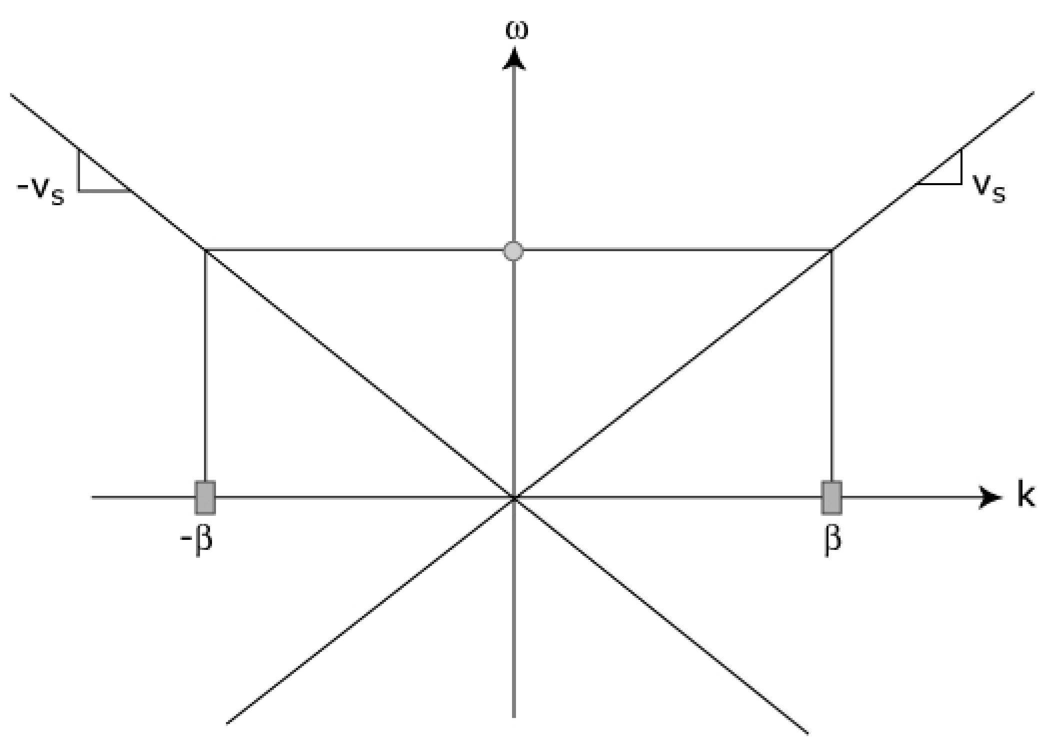
$$\not{m} \frac{\partial^2 \xi}{\partial t^2} = \frac{T}{m} \frac{\partial^2 \xi}{\partial x^2}$$

$$v_s = \sqrt{T/m}$$

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2} \quad \text{(wave equation)}$$

$$\xi = \mathrm{Re}\left[\hat{\xi}\, e^{j(\omega t - kx)}\right]$$

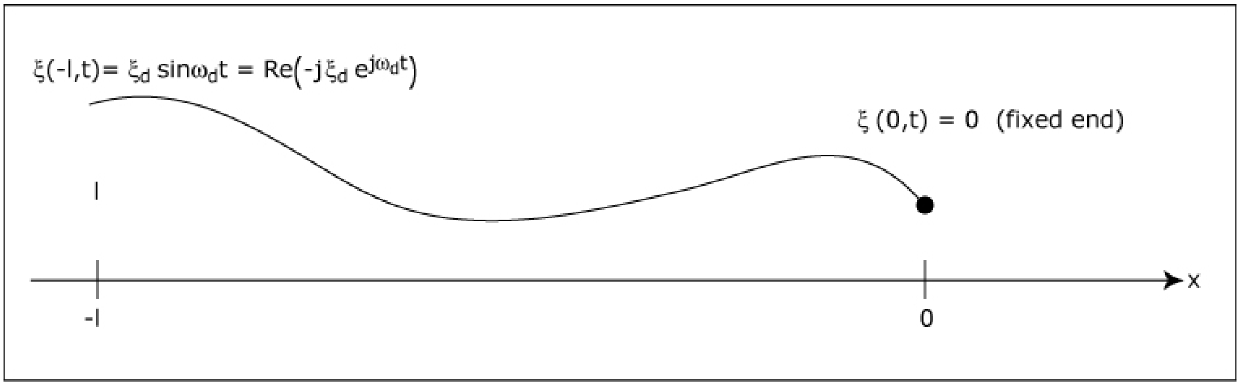
$$-\omega^2 \hat{\xi} = -k^2 v_s^2 \hat{\xi} \Rightarrow \omega^2 = k^2 v_s^2 \Rightarrow \omega = \pm k\, v_s$$




Dispersion equation for waves on the simple string.

B. Driven and Transient Responses

$$\xi(-l,t) = \xi_d \sin \omega_d t = \mathrm{Re}\left(-j\xi_d e^{j\omega_d t}\right)$$




$$k = \pm \frac{\omega_d}{v_s}$$

$$\xi(x,t) = \mathrm{Re}\left[\left(\hat{\xi}_1 e^{-\frac{j\omega_d}{v_s}x} + \hat{\xi}_2 e^{+j\frac{\omega_d}{v_s}x}\right)e^{j\omega_d t}\right]$$

$$\xi(0,t) = 0 = \mathrm{Re}\left[\left(\hat{\xi}_1 + \hat{\xi}_2\right)e^{j\omega_d t}\right] \Rightarrow \hat{\xi}_1 = -\hat{\xi}_2$$

$$\hat{\xi}(-l,t) = -j\xi_d = \hat{\xi}_1 e^{\frac{j\omega_d l}{v_s}} + \hat{\xi}_2 e^{-\frac{j\omega_d l}{v_s}} = \hat{\xi}_1\left(e^{\frac{j\omega_d l}{v_s}} - e^{-j\omega_d l/v_s}\right)$$

$$= 2j\hat{\xi}_1 \sin\frac{\omega_d l}{v_s}$$

$$-j\xi_d = 2j \sin\frac{\omega_d l}{v_s}\ \hat{\xi}_1 \Rightarrow \hat{\xi}_1 = -\frac{\xi_d}{2\sin\frac{\omega_d l}{v_s}}$$

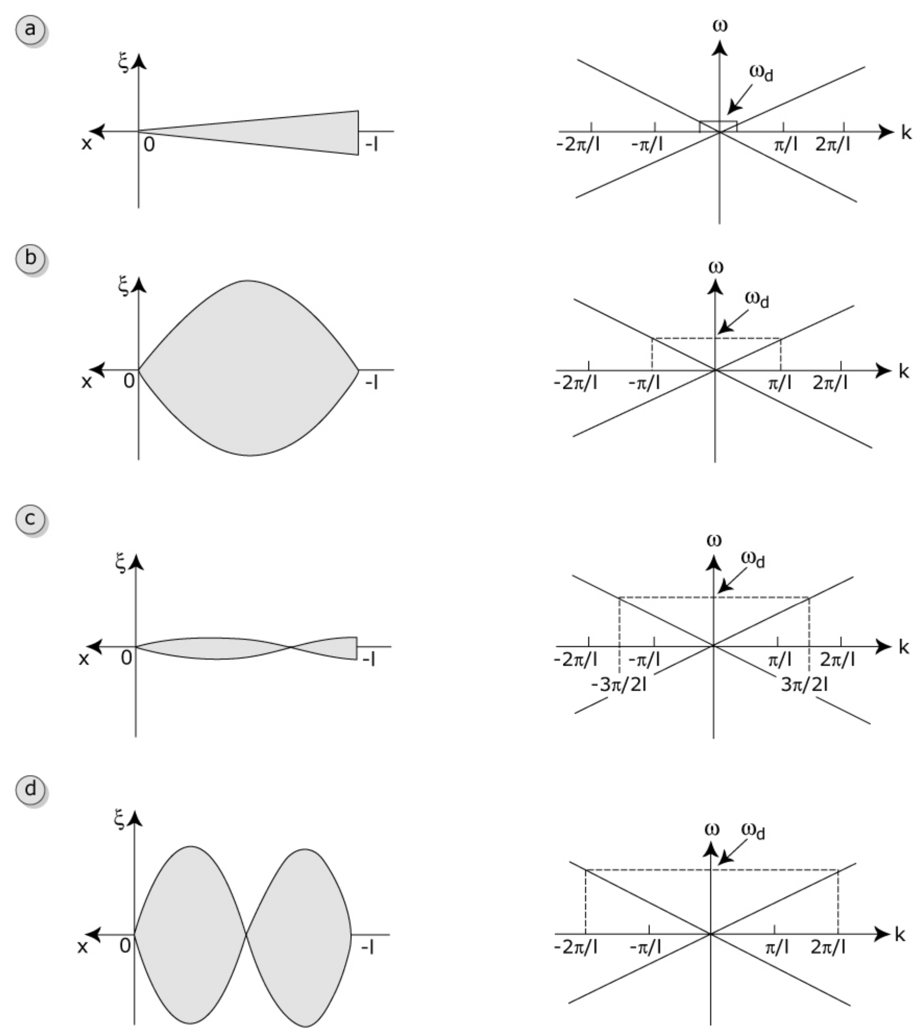
$$\hat{\xi}(x) = -\frac{\xi_d}{2\sin\frac{\omega_d l}{v_s}}\left(e^{\frac{-j\omega_d x}{v_s}} - e^{\frac{j\omega_d x}{v_s}}\right) = \frac{+\xi_d}{\cancel{2}\sin\frac{\omega_d l}{v_s}}\left(+\cancel{2}\, j\sin\frac{\omega_d x}{v_s}\right)$$

$$= \frac{j\sin\frac{\omega_d x}{v_s}\ \xi_d}{\sin\frac{\omega_d l}{v_s}}$$

$$\xi(x,t) = \operatorname{Re}\left[\hat{\xi}(x)e^{j\omega_d t}\right] = \operatorname{Re}\left[j\frac{\sin\frac{\omega_d x}{v_s}}{\sin\frac{\omega_d l}{v_s}}\xi_d e^{j\omega_d t}\right] = -\frac{\xi_d \sin\frac{\omega_d x}{v_s}}{\sin\frac{\omega_d l}{v_s}}\sin\omega_d t$$

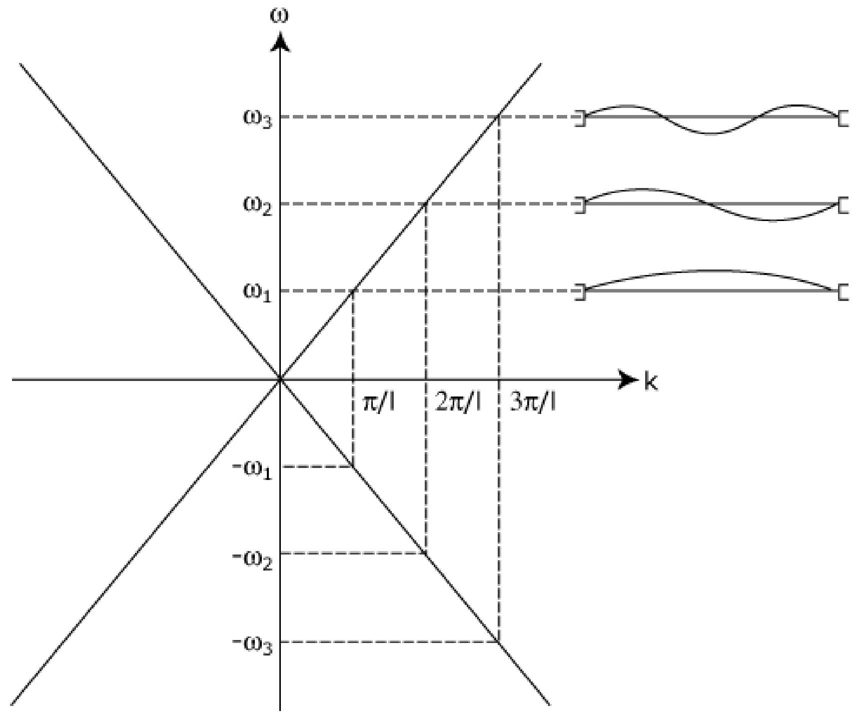
$$\text{Resonance: } \sin\frac{\omega_d l}{v_s} = 0$$

$$\frac{\omega_d l}{v_s} = n\pi \ , \quad n=1, 2, 3, \ldots.$$



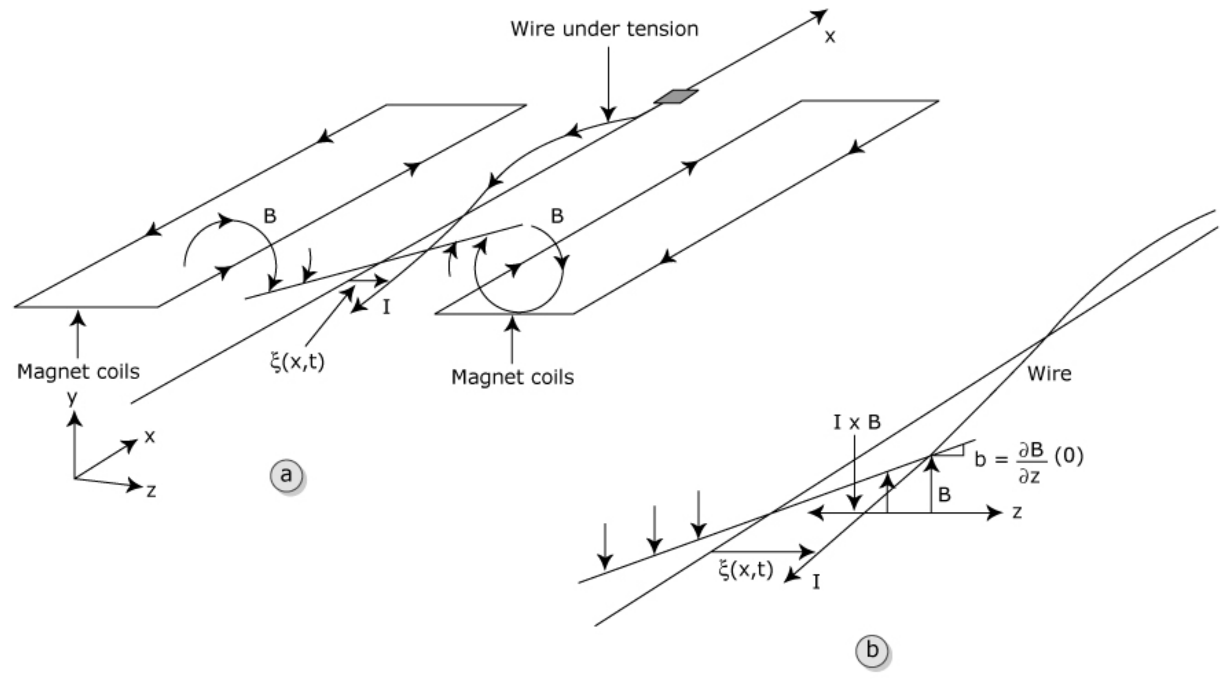

Sketch of experiment in which a taut spring is fixed at the left end and deflected sinusoidally at the right end. (a) Deflections in the quasi-static limit at which the frequency is low compared with the reciprocal of the time required for a disturbance to propagate from one end of the spring to the other; (b) to (d) deflection as frequency is varied from value at which $k=\pi/l$ to $k=2\pi/l$. The excitation amplitude is kept the same in going from (b) to (d). Actual experiment can be seen in film, "Complex Waves I" produced by Education Development Center for National Commitee on Electrical Engineering Films.

Simple Elastic Continua




Allowed wavenumbers (eigenvalues) $k = k_n$ as they are related to the eigen-frequencies $\omega_n$ by the dispersion equation.

## IV. Cut-off or Evanescent Waves




(a) A conducting wire is stretched along the x-axis and is free to undergo transverse motions in the horizontal plane. Magnet coils produce a field B which is zero along the x-axis; (b) the wire carries a current I so that deflections from the x-axis result in a force that tends to restore the wire to its equilibrium position.

$$m\frac{\partial^2 \xi}{\partial t^2} = T\frac{\partial^2 \xi}{\partial x^2} + F_{ext}$$

$$F_{ext} = \left(\bar{I} \times \bar{B}\right) \cdot \bar{i}_z = -I\ \bar{i}_x \times \left(\underbrace{\frac{\partial B_y}{\partial z}}_{b}\ \xi\,\bar{i}_y\right) \cdot \bar{i}_z$$

$$= -I b\, \xi$$

$$\not{m}\frac{\partial^2 \xi}{\partial t^2} = \frac{T}{m}\frac{\partial^2 \xi}{\partial x^2} - \frac{I b\, \xi}{m}$$

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2\frac{\partial^2 \xi}{\partial x^2} - \omega_c{}^2 \xi \quad ; \quad v_s{}^2 = \frac{T}{m},\ \omega_c{}^2 = \frac{I b}{m}$$

$$\xi = \mathrm{Re}\left[\hat{\xi}\ e^{j(\omega t - kx)}\right]$$

$$+\omega^2 = +k^2 v_s^2 + \omega_c^2 \Rightarrow k = \pm\sqrt{\omega^2 - \omega_c^2} / v_s$$

$$\xi(0,t) = 0, \quad \xi(-l,t) = \xi_d \sin \omega_d t$$

$$k = \pm|k_r|, \quad \omega_d > \omega_c \quad \left( |k_r| = \sqrt{\omega_d^2 - \omega_c^2} / v_s \right)$$

$$k = \pm j|k_i|, \quad \omega_d < \omega_c \quad \left( |k_i| = \sqrt{\omega_c^2 - \omega_d^2} / v_s \right)$$

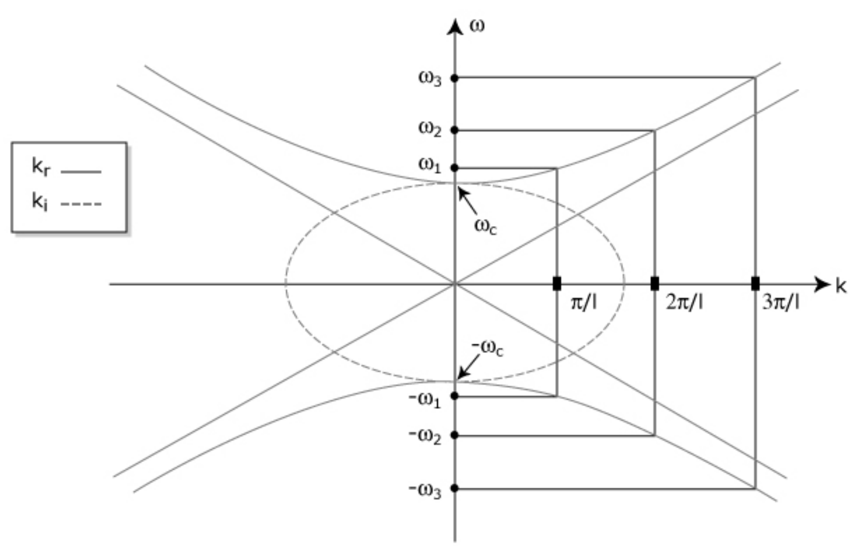
$\omega_d > \omega_c$

$$\xi = -\xi_d \frac{\sin|k_r|x}{\sin|k_r|l} \sin \omega_d t$$

$\omega_d < \omega_c$

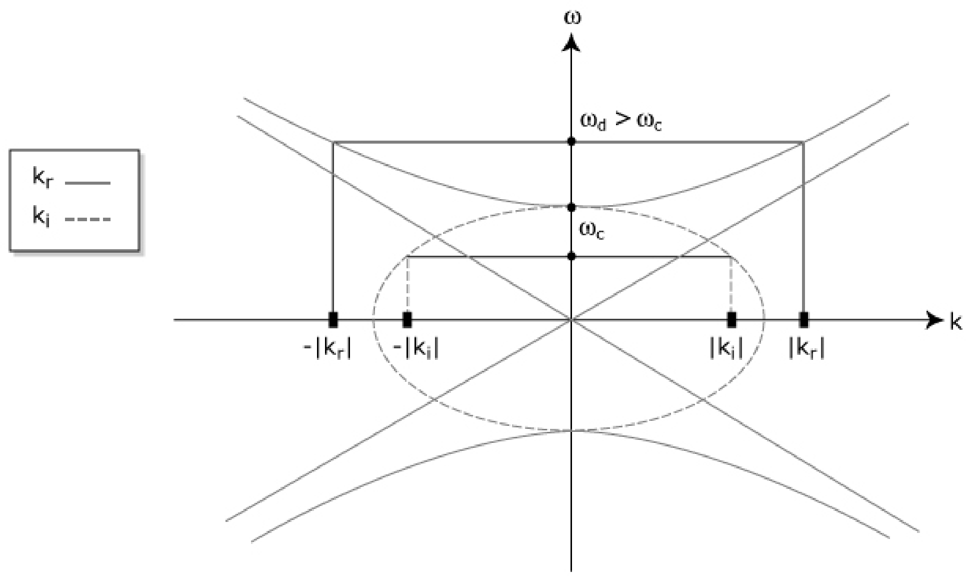
$$\xi = -\xi_d \frac{\sinh|k_i|x}{\sinh|k_i|l} \sin \omega_d t$$

Dynamics of Electromechanical Continuum




A dispersion equation for waves on the wire in (figure on page 8) showing the relationship between the eigenfrequencies $\omega_n$ and the eigenvalues $k = n\pi/l$.

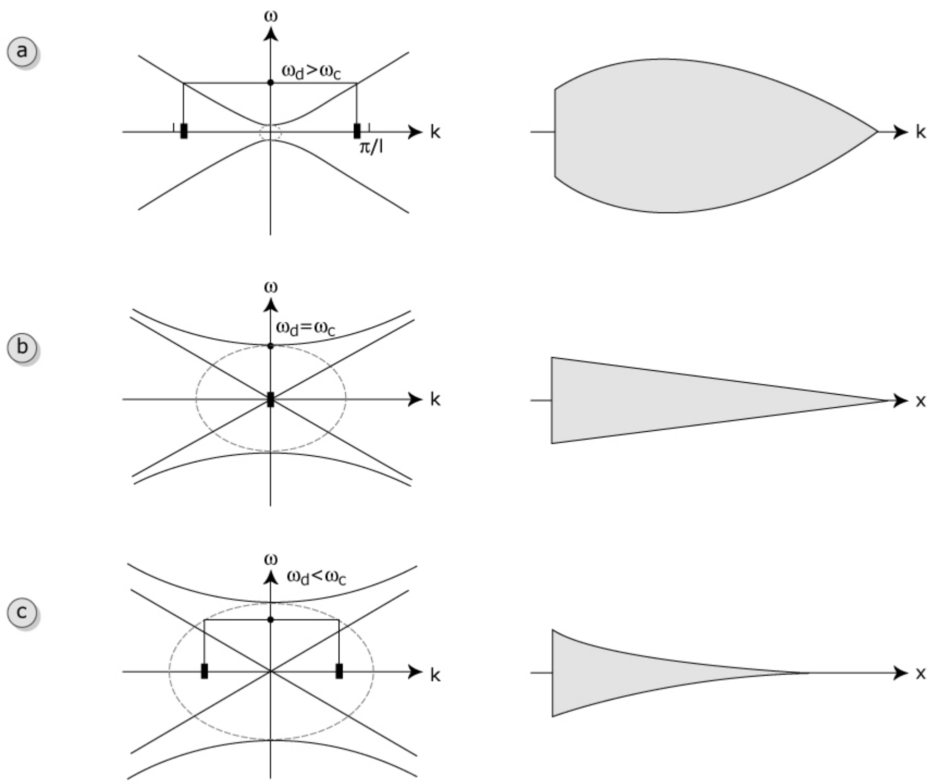
Waves and Instabilities in Stationary Media




Dispersion relation for the wire subject to a restoring force distributed along its length (for the case shown in figure on page 8). Complex values of k are shown as functions of real values of ω.

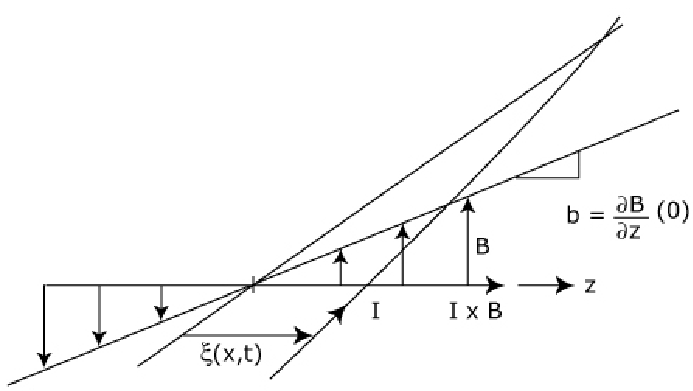
$$\text{resonance} \quad \left(|k_r| l = n\pi\right) \Rightarrow \frac{\omega^2 - \omega_c^2}{v_s^2} = \left(\frac{n\pi}{l}\right)^2 \Rightarrow \omega = \left[\omega_c^2 + \left(\frac{n\pi}{l}\right)^2 v_s^2\right]^{1/2}$$

Waves and Instabilities in Stationary Media




Envelope of wire deflection in magnetic field. The wire is fixed at the right end and driven at a fixed sinusoidal frequency at the left end. The ω-k plots show the effect of the current I on the dispersion equation. The current I (or cutoff frequency $\omega_c$) is being raised so that (a), $I \approx 0$, (b) I is sufficient just to cut off the propagation ($\omega_d = \omega_c$), and (c) the waves are evanescent, $\omega_d < \omega_c$. This experiment can be seen in the film "Complex Waves I," produced for the National Committee on Electrical Engineering films by Education Development Center, Newton, Mass.

## V. Absolute or Non convective Instability




Wire carrying current I in a magnetic field that is zero along the axis ξ=0. Current is reversed from the situation shown in figure on page 8.

$$F_{ext} = +Ib\xi$$

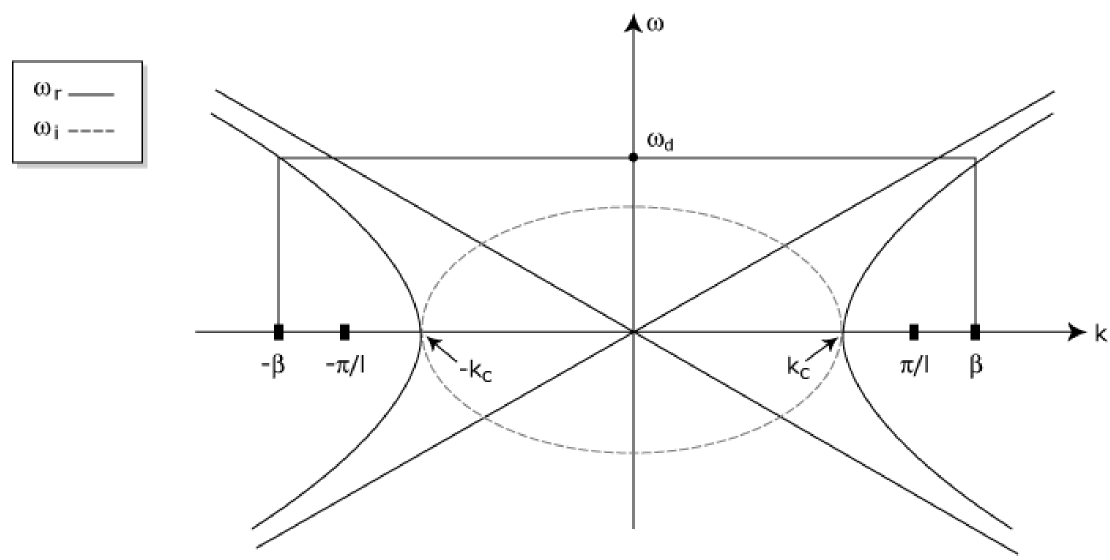
$$\cancel{m}\frac{\partial^2 \xi}{\partial t^2} = \frac{T}{m}\frac{\partial^2 \xi}{\partial x^2} + \frac{Ib\xi}{m}$$

$$\frac{\partial^2 \xi}{\partial t^2} = v_s^2 \frac{\partial^2 \xi}{\partial x^2} + \omega_c^2 \quad , \quad v_s^2 = \frac{T}{m} \quad , \quad \omega_c^2 = \frac{Ib}{m}$$

$$\xi = \mathrm{Re}\left[\hat{\xi}\, e^{j(\omega t - kx)}\right]$$

$$-\omega^2 = -k^2 v_s^2 + \omega_c^2 \Rightarrow \omega^2 = k^2 v_s^2 - \omega_c^2$$

Waves and Instabilities in Stationary Media




Plot of the dispersion equation for physical situation shown in (figure on page 8) with the current reversed, as in (figure on page 11). Complex values of ω are shown for real values of k.

Take undriven spring= $\xi(-l, t) = 0, \quad \xi(0, t) = 0$

$$\xi = \mathrm{Re}\left[\hat{\xi}(x)\, e^{j\omega t}\right]$$

$$\hat{\xi}(x) = \xi_1 e^{-jkx} + \xi_2 e^{+jkx}$$

$$K = \sqrt{\frac{\omega^2 + \omega_c^2}{v_s}}$$

$$\hat{\xi}(x=0) = 0 = \xi_1 + \xi_2$$

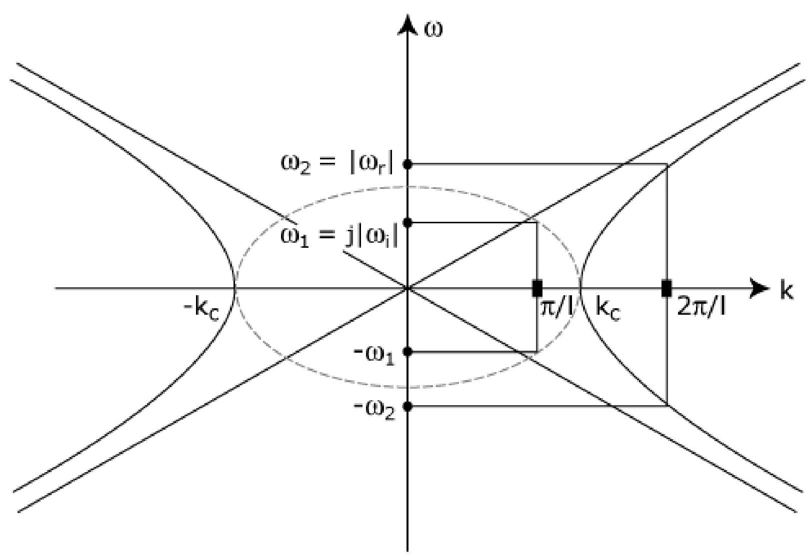
$$\hat{\xi}(x=-l) = 0 = \xi_1 e^{jkl} + \xi_2 e^{-jkl} = \xi_1 \left(e^{jkl} - e^{-jkl}\right)$$

$$= 2j\xi_1 \sin kl$$

$$k \not{l} = \frac{n\pi}{l}$$

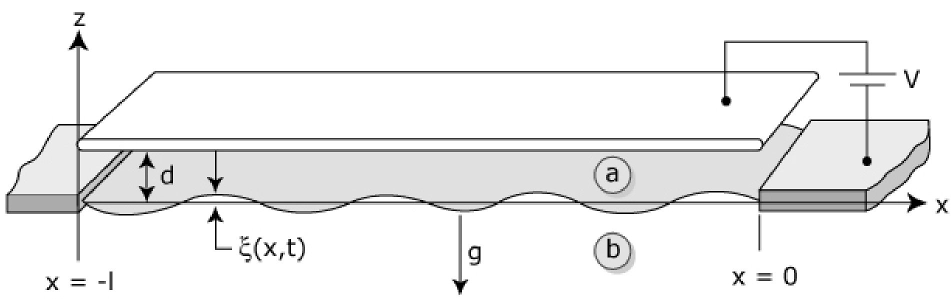
$$\omega^2 = \left(\frac{n\pi}{l}\right)^2 - \omega_c^2, \quad \text{if } \left(\frac{n\pi}{l}\right) < \omega_c, \ \omega = \pm j \left|\omega_i\right|$$

Negative imaginary roots are absolutely unstable: $\sim e^{|\omega_i| t}$




The dispersion equation for the system of (figure on page 8) with current as shown in figure on page 11. Complex values of ω are shown for real values of k. The allowed values of k give rise to the eigenfrequencies as shown.

## VI. Electric Field Levitation of Membrane




Conducting elastic membrane held horizontal in a gravitational field by an electrostatic force.

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \frac{\partial^2 \xi}{\partial x^2} - \sigma_{mg} + T_z^e$$

$$\bar{E} = \frac{-V}{d-\xi} \bar{i}_z$$

$$T_z^e = \left(T_{zj}^a - T_{zj}^b\right) n_j = T_{zz}^a = \frac{1}{2}\varepsilon_o E_z^2 = \frac{1}{2}\varepsilon_o \left(\frac{V}{d-\xi}\right)^2$$

$$= \frac{1}{2}\frac{\varepsilon_0 V^2}{d^2}\left(\frac{1}{1-\left(\xi/d\right)}\right)^2$$

$$\approx \frac{1}{2}\frac{\varepsilon_o V^2}{d^2}\left(1 + 2\,\xi/d\right)$$

$$\sigma_m \frac{\partial^2 \xi}{\partial t^2} = S \frac{\partial^2 \xi}{\partial x^2} - \sigma_{mg} + \frac{1}{2}\frac{\varepsilon_o V^2}{d^2} + \frac{\varepsilon_o V^2}{d^3}\xi$$

Equilibrium: $\xi = 0 \Rightarrow \sigma_m g = \frac{1}{2}\frac{\varepsilon_o V^2}{d^2}$

Perturbations : $\cancel{\sigma_m} \frac{\partial^2 \xi}{\partial t^2} = \frac{S}{\sigma_m} \frac{\partial^2 \xi}{\partial x^2} + \frac{\varepsilon_o V^2}{d^3 \sigma_m} \xi$

$$v_s^2 = \frac{S}{\sigma_m}, \qquad \frac{\varepsilon_o V^2}{d^3 \sigma_m} = \omega_c^2$$

$$\xi = \mathrm{Re}\left[\hat{\xi}\, e^{j(\omega t - kx)}\right]$$

$$-\omega^2 = -k^2 v_s^2 + \omega_c^2$$

$$\xi(0,t) = \xi(-l,t) = 0 \Rightarrow k = \frac{n\pi}{l}$$

$$\omega^2 = \left(\frac{n\pi}{l} v_s\right)^2 - \omega_c^2$$

Stable if: $\omega_c^2 = \frac{\varepsilon_o V^2}{\sigma_m d^3} < \left(\frac{n\pi}{l} v_s\right)^2 \Rightarrow \frac{\varepsilon_o V^2}{\sigma_m d^3 v_s^2} = \frac{\varepsilon_o V^2}{d^3 s} < \left(\frac{n\pi}{l}\right)^2$

First Unstable mode: $n = 1$

$$\underbrace{\frac{\varepsilon_o V^2}{d^3 S} < \left(\frac{\pi}{l}\right)^2}$$

$$\frac{2\sigma_m g}{dS} < \left(\frac{\pi}{l}\right)^2 \Rightarrow \sigma_m < \frac{\left(\frac{\pi}{l}\right)^2 S\, d}{2g}$$