Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.685 Electric Machines

Class Notes 7: Permanent Magnet "Brushless DC" Motors September 5, 2005

1 Introduction

This document is a brief introduction to the design evaluation of permanent magnet motors, with an eye toward servo and drive applications. It is organized in the following manner: First, we describe three different geometrical arrangements for permanent magnet motors:

1. Surface Mounted Magnets, Conventional Stator,
2. Surface Mounted Magnets, Air-Gap Stator Winding, and
3. Internal Magnets (Flux Concentrating).

After a qualitative discussion of these geometries, we will discuss the elementary rating parameters of the machine and show how to arrive at a rating and how to estimate the torque and power vs. speed capability of the motor. Then we will discuss how the machine geometry can be used to estimate both the elementary rating parameters and the parameters used to make more detailed estimates of the machine performance.

Some of the more involved mathematical derivations are contained in appendices to this note.

2 Motor Morphologies

There are, of course, many ways of building permanent magnet motors, but we will consider only a few in this note. Actually, once these are understood, rating evaluations of most other geometrical arrangements should be fairly straightforward. It should be understood that the "rotor inside" vs. "rotor outside" distinction is in fact trivial, with very few exceptions, which we will note.

2.1 Surface Magnet Machines

Figure 1 shows the basic *magnetic* morphology of the motor with magnets mounted on the surface of the rotor and an otherwise conventional stator winding. This sketch does not show some of the important mechanical aspects of the machine, such as the means for fastening the permanent magnets to the rotor, so one should look at it with a bit of caution. In addition, this sketch and the other sketches to follow are not necessarily to a scale that would result in workable machines.

This figure shows an axial section of a four-pole ($p = 2$) machine. The four magnets are mounted on a cylindrical rotor "core", or shaft, made of ferromagnetic material. Typically this would simply be a steel shaft. In some applications the magnets may be simply bonded to the steel. For applications in which a glue joint is not satisfactory (e.g. for high speed machines) some sort of rotor banding or retaining ring structure is required.

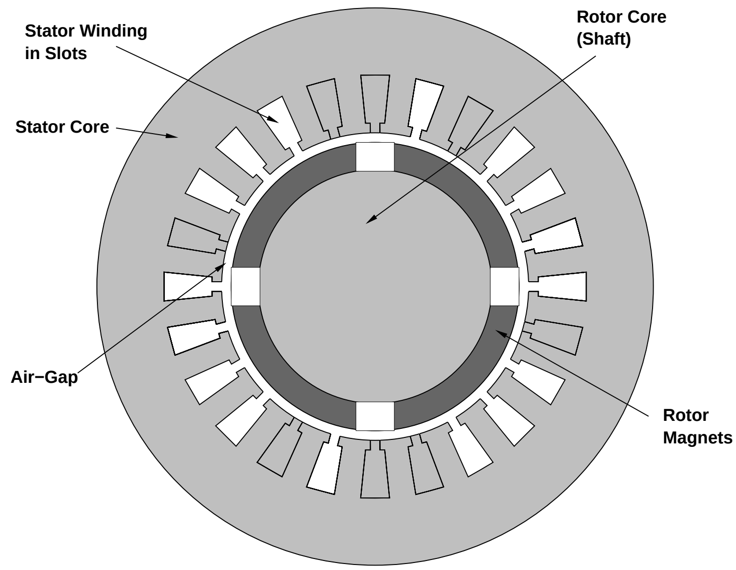

Figure 1: Axial View of a Surface Mount Motor

The stator winding of this machine is "conventional", very much like that of an induction motor, consisting of wires located in slots in the surface of the stator core. The stator core itself is made of laminated ferromagnetic material (probably silicon iron sheets), the character and thickness of the sheets determined by operating frequency and efficiency requirements. They are required to carry alternating magnetic fields, so must be laminated to reduce eddy current losses.

This sort of machine is simple in construction. Note that the operating magnetic flux density in the air-gap is nearly the same as in the magnets, so that this sort of machine cannot have air-gap flux densities higher than that of the remanent flux density of the magnets. If low cost ferrite magnets are used, this means relatively low induction and consequently relatively low efficiency and power density. (Note the qualifier "relatively" here!). Note, however, that with modern, high performance permanent magnet materials in which remanent flux densities can be on the order of 1.2 T, air-gap working flux densities can be on the order of 1 T. With the requirement for slots to carry the armature current, this may be a practical limit for air-gap flux density anyway.

It is also important to note that the magnets in this design are really in the "air gap" of the machine, and therefore are exposed to all of the time- and space- harmonics of the stator winding MMF. Because some permanent magnets have electrical conductivity (particularly the higher performance magnets), any asynchronous fields will tend to produce eddy currents and consequent losses in the magnets.

2.2 Interior Magnet or Flux Concentrating Machines

Interior magnet designs have been developed to counter several apparent or real shortcomings of surface mount motors:

- Flux concentrating designs allow the flux density in the air-gap to be higher than the flux density in the magnets themselves.

- In interior magnet designs there is some degree of shielding of the magnets from high order space harmonic fields by the pole pieces.
- There are control advantages to some types of interior magnet motors, as we will show anon. Essentially, they have relatively large negative saliency which enhances "flux weakening" for high speed operation, in rather direct analogy to what is done in DC machines.
- Some types of internal magnet designs have (or claim) structural advantages over surface mount magnet designs.

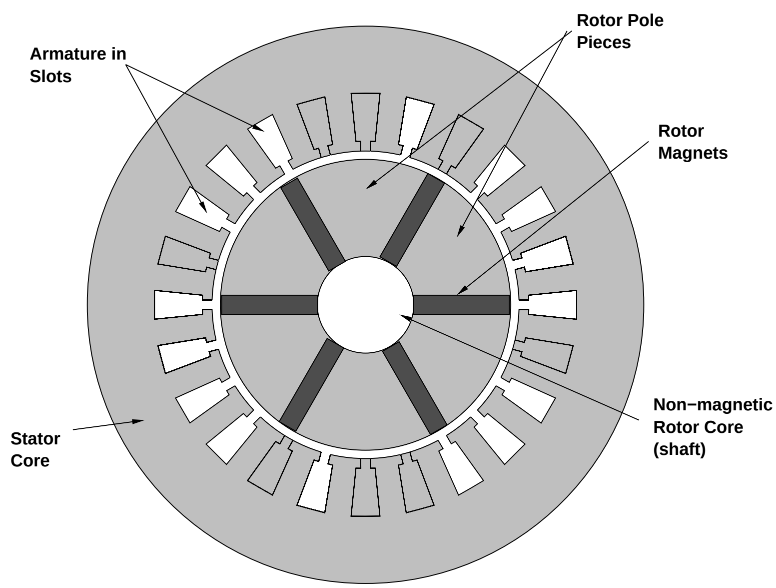

Figure 2: Axial View of a Flux Concentrating Motor

The geometry of one type of internal magnet motor is shown (crudely) in Figure 2. The permanent magnets are oriented so that their magnetization is azimuthal. They are located between wedges of magnetic material (the pole pieces) in the rotor. Flux passes through these wedges, going radially at the air- gap, then azimuthally through the magnets. The central core of the rotor must be non-magnetic, to prevent "shorting out" the magnets. No structure is shown at all in this drawing, but quite obviously this sort of rotor is a structural challenge. Shown is a six-pole machine. Typically, one does not expect flux concentrating machines to have small pole numbers, because it is difficult to get more area inside the rotor than around the periphery. On the other hand, a machine built in this way but without substantial flux concentration will still have saliency and magnet shielding properties.

A second morphology for an internal magnet motor is shown in Figure 3. This geometry has been proposed for highly salient synchronous machines without permanent magnets: such machines would run on the saliency torque and are called *synchronous reluctance* motors. however, the saliency slots may be filled with permanent magnet material, giving them some internally generated flux as well. The rotor iron tends to short out the magnets, so that the 'bridges' around the ends of the permanent magnets must be relatively thin. They are normally saturated.

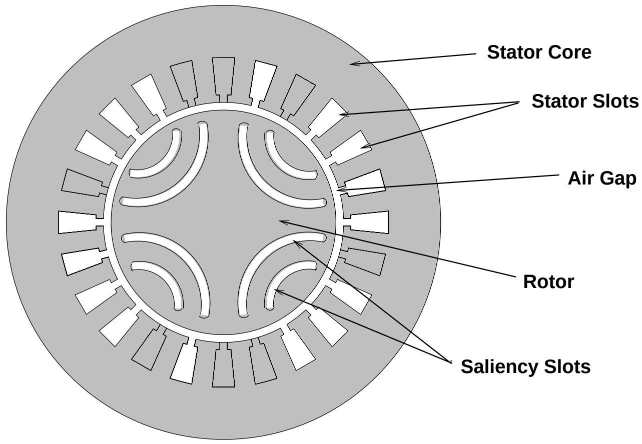

Figure 3: Axial View of Internal Magnet Motor

At first sight, these machines appear to be quite complicated to analyze, and that judgement seems to hold up.

2.3 Air Gap Armature Windings

Shown in Figure 4 is a surface-mounted magnet machine with an air-gap, or surface armature winding. Such machines take advantage of the fact that modern permanent magnet materials have very low permeabilities and that, therefore, the magnetic field produced is relatively insensitive to the size of the air-gap of the machine. It is possible to eliminate the stator teeth and use all of the periphery of the air-gap for windings.

Not shown in this figure is the structure of the armature winding. This is not an issue in "conventional" stators, since the armature is contained in slots in the iron stator core. The use of an air-gap winding gives opportunities for economy of construction, new armature winding forms such as helical windings, elimination of "cogging" torques, and (possibly) higher power densities.

3 Zeroth Order Rating

In determining the rating of a machine, we may consider two separate sets of parameters. The first set, the elementary rating parameters, consist of the machine inductances, internal flux linkage and stator resistance. From these and a few assumptions about base and maximum speed it is possible to get a first estimate of the rating and performance of the motor. More detailed performance estimates, including efficiency in sustained operation, require estimation of other parameters. We will pay more attention to that first set of parameters, but will attempt to show how at least some of the more complete operating parameters can be estimated.

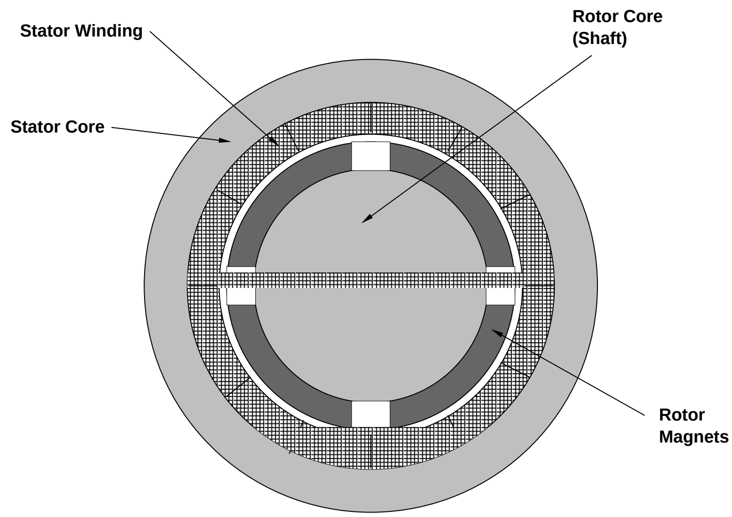

Figure 4: Axial View of a PM Motor With an Air-Gap Winding

3.1 Voltage and Current: Round Rotor

To get started, consider the equivalent circuit shown in Figure 5. This is actually the equivalent circuit which describes all *round rotor* synchronous machines. It is directly equivalent only to some of the machines we are dealing with here, but it will serve to illustrate one or two important points.

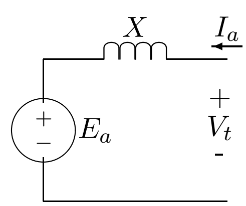

Figure 5: Synchronous Machine Equivalent Circuit

What is shown here is the equivalent circuit of a single phase of the machine. Most motors are three-phase, but it is not difficult to carry out most of the analysis for an arbitrary number of phases. The circuit shows an internal voltage E_a and a reactance X which together with the terminal current I determine the terminal voltage V. In this picture armature resistance is ignored. If the machine is running in the sinusoidal steady state, the major quantities are of the form:

$$
\begin{aligned}
E_a &= \omega\lambda_a \cos(\omega t + \delta) \\
V_t &= V \cos \omega t
\end{aligned}
$$

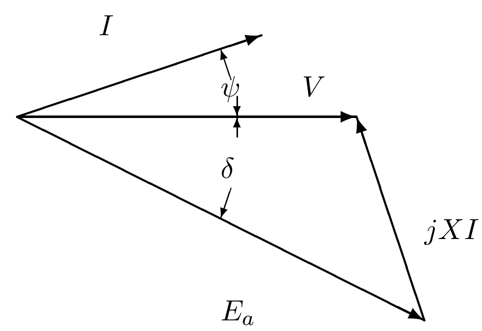

Figure 6: Phasor Diagram For A Synchronous Machine

$$I_a = I\cos(\omega t - \psi)$$

The machine is in synchronous operation if the internal and external voltages are at the same frequency and have a constant (or slowly changing) phase relationship (δ). The relationship between the major variables may be visualized by the phasor diagram shown in Figure 3.1. The internal voltage is just the time derivative of the internal flux from the permanent magnets, and the voltage drop in the machine reactance is also the time derivative of flux produced by armature current in the air-gap and in the "leakage" inductances of the machine. By convention, the angle ψ is positive when current I lags voltage V and the angle δ is positive then internal voltage E_a leads terminal voltage V. So both of these angles have negative sign in the situation shown in Figure 3.1.

If there are q phases, the *time average* power produced by this machine is simply:

$$P = \frac{q}{2} V I \cos\psi$$

For most polyphase machines operating in what is called "balanced" operation (all phases doing the same thing with uniform phase differences between phases), torque (and consequently power) are approximately constant. Since we have ignored power dissipated in the machine armature, it must be true that power absorbed by the internal voltage source is the same as terminal power, or:

$$P = \frac{q}{2} E_a I \cos(\psi - \delta)$$

Since in the steady state:

$$P = \frac{\omega}{p} T$$

where T is torque and ω/p is mechanical rotational speed, torque can be derived from the terminal quantities by simply:

$$T = p\frac{q}{2}\lambda_a I \cos(\psi - \delta)$$

In principal, then, to determine the torque and hence power rating of a machine it is only necessary to determine the internal flux, the terminal current capability, and the speed capability of the rotor. In fact it is *almost* that simple. Unfortunately, the model shown in Figure 5 is not quite complete for some of the motors we will be dealing with, and we must go one more level into machine theory.

3.2 A Little Two-Reaction Theory

The material in this subsection is framed in terms of three-phase ($q = 3$) machine theory, but it is actually generalizable to an arbitrary number of phases. Suppose we have a machine whose three-phase armature can be characterized by *internal* fluxes and inductance which may, in general, not be constant but is a function of rotor position. Note that the simple model we presented in the previous subsection does not conform to this picture, because it assumes a constant terminal inductance. In that case, we have:

$$\underline{\lambda}_{ph} = \underline{\underline{L}}_{ph}\underline{I}_{ph} + \underline{\lambda_R} \tag{1}$$

where $\underline{\lambda_R}$ is the set of internally produced fluxes (from the permanent magnets) and the stator winding may have both self- and mutual- inductances.

Now, we find it useful to do a transformation on these stator fluxes in the following way: each armature quantity, including flux, current and voltage, is projected into a coordinate system that is fixed to the rotor. This is often called the *Park's Transformation.* For a three phase machine it is:

$$\begin{bmatrix} u_d \\ u_q \\ u_0 \end{bmatrix} = \underline{u}_{dq} = \underline{\underline{T}}\underline{u}_{ph} = \underline{\underline{T}} \begin{bmatrix} u_a \\ u_b \\ u_c \end{bmatrix} \tag{2}$$

Where the transformation and its inverse are:

$$\underline{\underline{T}} = \frac{2}{3} \begin{bmatrix} \cos\theta & \cos(\theta - \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) \\ -\sin\theta & -\sin(\theta - \frac{2\pi}{3}) & -\sin(\theta + \frac{2\pi}{3}) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{3}$$

$$\underline{\underline{T}}^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 1 \\ \cos(\theta - \frac{2\pi}{3}) & -\sin(\theta - \frac{2\pi}{3}) & 1 \\ \cos(\theta + \frac{2\pi}{3}) & -\sin(\theta + \frac{2\pi}{3}) & 1 \end{bmatrix} \tag{4}$$

It is easy to show that balanced polyphase quantities in the stationary, or phase variable frame, translate into *constant* quantities in the so-called "d-q" frame. For example:

$$\begin{aligned} I_a &= I\cos\omega t \\ I_b &= I\cos(\omega t - \frac{2\pi}{3}) \\ I_c &= I\cos(\omega t + \frac{2\pi}{3}) \\ \theta &= \omega t + \theta_0 \end{aligned}$$

maps to:

$$\begin{aligned} I_d &= I\cos\theta_0 \\ I_q &= -I\sin\theta_0 \end{aligned}$$

Now, if $\theta = \omega t + \theta_0$, the transformation coordinate system is chosen correctly and the "d-" axis will correspond with the axis on which the rotor magnets are making positive flux. That happens if, when $\theta = 0$, phase A is linking maximum positive flux from the permanent magnets. If this is the case, the *internal* fluxes are:

$$\begin{aligned}
\lambda_{aa} &= \lambda_f \cos\theta \\
\lambda_{ab} &= \lambda_f \cos(\theta - \frac{2\pi}{3}) \\
\lambda_{ac} &= \lambda_f \cos(\theta + \frac{2\pi}{3})
\end{aligned}$$

Now, if we compute the fluxes in the d-q frame, we have:

$$\underline{\lambda}_{dq} = \underline{\underline{L}}_{dq} \underline{I}_{dq} + \underline{\lambda}_R = \underline{\underline{T}} \underline{\underline{L}}_{ph} \underline{\underline{T}}^{-1} \underline{I}_{dq} + \underline{\lambda}_R \tag{5}$$

Now: two things should be noted here. The first is that, if the coordinate system has been chosen as described above, the flux induced by the rotor is, in the d-q frame, simply:

$$\underline{\lambda}_R = \begin{bmatrix} \lambda_f \\ 0 \\ 0 \end{bmatrix} \tag{6}$$

That is, the magnets produce flux *only* on the d- axis.

The second thing to note is that, under certain assumptions, the inductances in the d-q frame are *independent of rotor position* and have no mutual terms. That is:

$$\underline{\underline{L}}_{dq} = \underline{\underline{T}} \underline{\underline{L}}_{ph} \underline{\underline{T}}^{-1} = \begin{bmatrix} L_d & 0 & 0 \\ 0 & L_q & 0 \\ 0 & 0 & L_0 \end{bmatrix} \tag{7}$$

The assertion that inductances in the d-q frame are constant is actually questionable, but it is close enough to being true and analyses that use it have proven to be close enough to being correct that it (the assertion) has held up to the test of time. In fact the deviations from independence on rotor position are small. Independence of axes (that is, absence of mutual inductances in the d-q frame) is correct because the two axes are physically orthogonal. We tend to ignore the third, or "zero" axis in this analysis. It doesn't couple to anything else and has neither flux nor current anyway. Note that the direct- and quadrature- axis inductances are in principle straightforward to compute. They are

direct axis the inductance of one of the armature phases (corrected for the fact of multiple phases) with the rotor aligned with the axis of the phase, and

quadrature axis the inductance of one of the phases with the rotor aligned 90 electrical degrees away from the axis of that phase.

Next, armature voltage is, ignoring resistance, given by:

$$\underline{V}_{ph} = \frac{d}{dt} \underline{\lambda}_{ph} = \frac{d}{dt} \underline{\underline{T}}^{-1} \underline{\lambda}_{dq} \tag{8}$$

and that the *transformed* armature voltage must be:

$$\begin{aligned}
\underline{V}_{dq} &= \underline{\underline{T}}\underline{V}_{ph} \\
&= \underline{\underline{T}}\frac{d}{dt}(\underline{\underline{T}}^{-1}\underline{\lambda}_{dq}) \\
&= \frac{d}{dt}\underline{\lambda}_{dq} + (\underline{\underline{T}}\frac{d}{dt}\underline{\underline{T}}^{-1})\underline{\lambda}_{dq} \qquad (9)
\end{aligned}$$

The second term in this expresses "speed voltage". A good deal of straightforward but tedious manipulation yields:

$$\underline{\underline{T}}\frac{d}{dt}\underline{\underline{T}}^{-1} = \begin{bmatrix} 0 & -\frac{d\theta}{dt} & 0 \\ \frac{d\theta}{dt} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (10)$$

The direct- and quadrature- axis voltage expressions are then:

$$V_d = \frac{d\lambda_d}{dt} - \omega\lambda_q \qquad (11)$$

$$V_q = \frac{d\lambda_q}{dt} + \omega\lambda_d \qquad (12)$$

where

$$\omega = \frac{d\theta}{dt}$$

Instantaneous *power* is given by:

$$P = V_a I_a + V_b I_b + V_c I_c \qquad (13)$$

Using the transformations given above, this can be shown to be:

$$P = \frac{3}{2}V_d I_d + \frac{3}{2}V_q I_q + 3V_0 I_0 \qquad (14)$$

which, in turn, is:

$$P = \omega\frac{3}{2}(\lambda_d I_q - \lambda_q I_d) + \frac{3}{2}(\frac{d\lambda_d}{dt}I_d + \frac{d\lambda_q}{dt}I_q) + 3\frac{d\lambda_0}{dt}I_0 \qquad (15)$$

Then, noting that $\omega = p\Omega$ and that (15) describes electrical terminal power as the sum of shaft power and rate of change of stored energy, we may deduce that torque is given by:

$$T = \frac{q}{2}p(\lambda_d I_q - \lambda_q I_d) \qquad (16)$$

Note that we have stated a generalization to a q- phase machine even though the derivation given here was carried out for the $q = 3$ case. Of course three phase machines are by far the most common case. Machines with higher numbers of phases behave in the same way (and this generalization is valid for all purposes to which we put it), but there are more rotor variables analogous to "zero axis".

Now, noting that, in general, L_d and L_q are not necessarily equal,

$$\lambda_d = L_d I_d + \lambda_f \qquad (17)$$

$$\lambda_q = L_q I_q \qquad (18)$$

then torque is given by:

$$T = p\frac{q}{2}\left(\lambda_f + \left(L_d - L_q\right) I_d\right) I_q \qquad (19)$$

3.3 Finding Torque Capability

For high performance drives, we will generally assume that the power supply, generally an inverter, can supply currents in the correct spatial relationship to the rotor to produce torque in some reasonably effective fashion. We will show in this section how to determine, given a required torque (or if the torque is limited by either voltage or current which we will discuss anon), what the values of I_d and I_q must be. Then the power supply, given some means of determining where the rotor is (the instantaneous value of θ), will use the inverse Park's transformation to determine the instantaneous valued required for phase currents. This is the essence of what is known as "field oriented control", or putting stator currents in the correct location *in space* to produce the required torque.

Our objective in this section is, given the elementary parameters of the motor, find the capability of the motor to produce torque. There are three things to consider here:

- Armature current is limited, generally by heating,
- A second limit is the voltage capability of the supply, particularly at high speed, and
- If the machine is operating within these two limits, we should consider the optimal placement of currents (that is, how to get the most torque per unit of current to minimize losses).

Often the discussion of current placement is carried out using, as a tool to visualize what is going on, the I_d, I_q plane. Operation in the steady state implies a single point on this plane. A simple illustration is shown in Figure 7. The thermally limited armature current capability is represented as a circle around the origin, since the magnitude of armature current is just the length of a vector from the origin in this space. In general, for permanent magnet machines with buried magnets, $L_d < L_q$, so the optimal operation of the machine will be with negative I_d. We will show how to determine this optimum operation anon, but it will in general follow a curve in the I_d, I_q plane as shown.

Finally, an ellipse describes the *voltage* limit. To start, consider what would happen if the terminals of the machine were to be short-circuited so that $V = 0$. If the machine is operating at sufficiently high speed so that armature resistance is negligible, armature current would be simply:

$$I_d = -\frac{\lambda_f}{L_d}$$

$$I_q = 0$$

Now, loci of constant flux turn out to be ellipses around this point on the plane. Since terminal flux is proportional to voltage and inversely proportional to frequency, if the machine is operating with a given terminal voltage, the ability of that voltage to command current in the I_d, I_q plane is an ellipse whose size "shrinks" as speed increases.

To simplify the mathematics involved in this estimation, we normalize reactances, fluxes, currents and torques. First, let us define the *base* flux to be simply $\lambda_b = \lambda_f$ and the *base* current I_b to be the armature capability. Then we define two *per-unit* reactances:

$$x_d = \frac{L_d I_b}{\lambda_b} \quad (20)$$

$$x_q = \frac{L_q I_b}{\lambda_b} \quad (21)$$

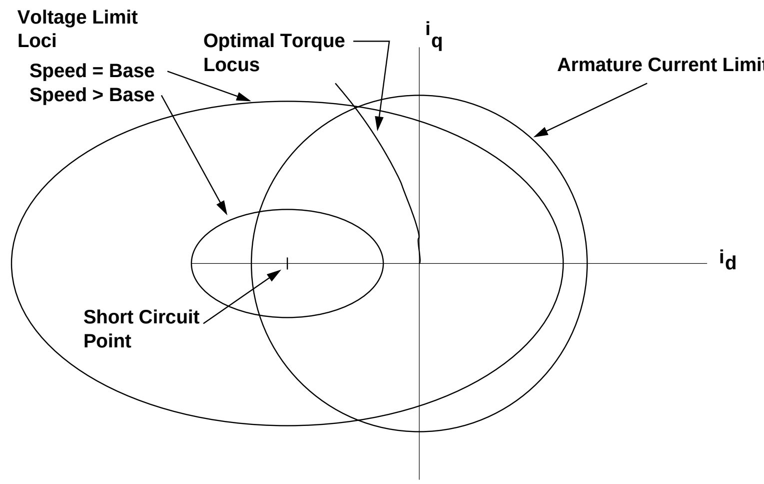

Figure 7: Limits to Operation

Next, define the *base torque* to be:

$$T_b = p\frac{q}{2}\lambda_b I_b$$

and then, given *per-unit* currents i_d and i_q, the *per-unit* torque is simply:

$$t_e = \left(1 - \left(x_q - x_d\right) i_d\right) i_q \tag{22}$$

It is fairly straightforward (but a bit tedious) to show that the locus of current-optimal operation (that is, the largest torque for a given current magnitude or the smallest current magnitude for a given torque) is along the curve:

$$i_d = -\sqrt{\frac{i_a^2}{2} + 2\left(\frac{1}{4\left(x_q - x_d\right)}\right)^2 - \frac{1}{2\left(x_q - x_d\right)}\sqrt{\left(\frac{1}{4\left(x_q - x_d\right)}\right)^2 + \frac{i_a^2}{2}}} \tag{23}$$

$$i_q = -\sqrt{\frac{i_a^2}{2} - 2\left(\frac{1}{4\left(x_q - x_d\right)}\right)^2 + \frac{1}{2\left(x_q - x_d\right)}\sqrt{\left(\frac{1}{4\left(x_q - x_d\right)}\right)^2 + \frac{i_a^2}{2}}} \tag{24}$$

The "rating point" will be the point along this curve when $i_a = 1$, or where this curve crosses the armature capability circle in the i_d, i_q plane. It should be noted that this set of expressions only works for salient machines. For non-salient machines, of course, torque-optimal current is on the q-axis. In general, for machines with saliency, the "per-unit" torque will *not* be unity at the rating, so that the rated, or "Base Speed" torque is not the "Base" torque, but:

$$T_r = T_b \times t_e \tag{25}$$

where t_e is calculated at the rating point (that is, $i_a = 1$ and i_d and i_q as per (23) and (24)).

For sufficiently low speeds, the power electronic drive can command the optimal current to produce torque up to rated. However, for speeds higher than the "Base Speed", this is no longer true. Define a per-unit terminal flux:

$$\psi = \frac{V}{\omega \lambda_b}$$

Operation at a given flux magnitude implies:

$$\psi^2 = (1 + x_d i_d)^2 + (x_q i_q)^2$$

which is an ellipse in the i_d, i_q plane. The *Base Speed* is that speed at which this ellipse crosses the point where the optimal current curve crosses the armature capability. Operation at the highest attainable torque (for a given speed) generally implies d-axis currents that are higher than those on the optimal current locus. What is happening here is the (negative) d-axis current serves to reduce effective machine flux and hence voltage which is limiting q-axis current. Thus operation above the base speed is often referred to as "flux weakening".

The strategy for picking the correct trajectory for current in the i_d, i_q plane depends on the value of the per-unit reactance x_d. For values of $x_d > 1$, it is possible to produce *some* torque at *any* speed. For values of $x_d < 1$, there is a speed for which no point in the armature current capability is within the voltage limiting ellipse, so that useful torque has gone to zero. Generally, the maximum torque operating point is the intersection of the armature current limit and the voltage limiting ellipse:

$$i_d = \frac{x_d}{x_q^2 - x_d^2} - \sqrt{\left(\frac{x_d}{x_q^2 - x_d^2}\right)^2 + \frac{x_q^2 - \psi^2 + 1}{x_q^2 - x_d^2}} \qquad (26)$$

$$i_q = \sqrt{1 - i_d^2} \qquad (27)$$

It may be that there is no intersection between the armature capability and the voltage limiting ellipse. If this is the case and if $x_d < 1$, torque capability at the given speed is zero.

If, on the other hand, $x_d > 1$, it may be that the intersection between the voltage limiting ellipse and the armature current limit is *not* the maximum torque point. To find out, we calculate the maximum torque point on the voltage limiting ellipse. This is done in the usual way by differentiating torque with respect to i_d while holding the relationship between i_d and i_q to be on the ellipse. The algebra is a bit messy, and results in:

$$i_d = -\frac{3x_d (x_q - x_d) - x_d^2}{4x_d^2 (x_q - x_d)} - \sqrt{\left(\frac{3x_d (x_q - x_d) - x_d^2}{4x_d^2 (x_q - x_d)}\right)^2 + \frac{(x_q - x_d)(\psi^2 - 1) + x_d}{2 (x_q - x_d) x_d^2}} \qquad (28)$$

$$i_q = \frac{1}{x_q}\sqrt{\psi^2 - (1 + x_d i_d)^2} \qquad (29)$$

Ordinarily, it is probably easiest to compute (28) and (29) first, then test to see if the currents are outside the armature capability, and if they are, use (26) and (27).

These expressions give us the capability to estimate the torque-speed curve for a machine. As an example, the machine described by the parameters cited in Table 1 is a (nominal) 3 HP, 4-pole, 3000 RPM machine.

The rated operating point turns out to have the following attributes:

Table 1: Example Machine

D- Axis Inductance	2.53 mHy
Q- Axis Inductance	6.38 mHy
Internal Flux	58.1 mWb
Armature Current	30 A

Table 2: Operating Characteristics of Example Machine

Per-Unit D-Axis Current At Rating Point	i_d	-.5924
Per-Unit Q-Axis Current At Rating Point	i_q	.8056
Per-Unit D-Axis Reactance	x_d	1.306
Per-Unit Q-Axis Reactance	x_q	3.294
Rated Torque (Nm)	T_r	9.17
Terminal Voltage at Base Point (V)		97

The loci of operation in the I_d, I_q plane is shown in Figure 8. The armature current limit is shown only in the second and third quadrants, so shows up as a semicircle. The two ellipses correspond with the rated point (the larger ellipse) and with a speed that is three times rated (9000 RPM). The torque-optimal current locus can be seen running from the origin to the rating point, and the higher speed operating locus follows the armature current limit. Figure 9 shows the torque/speed and power/speed curves. Note that this sort of machine only approximates “constant power” operation at speeds above the “base” or rating point speed.

4 Parameter Estimation

We are now at the point of estimating the major parameters of the motors. Because we have a number of different motor geometries to consider, and because they share parameters in not too orderly a fashion, this section will have a number of sub-parts. First, we calculate flux linkage, then reactance.

4.1 Flux Linkage

Given a machine which may be considered to be uniform in the axial direction, flux linked by a single, full-pitched coil which spans an angle from zero to π/p, is:

$$\phi = \int_0^{\frac{\pi}{p}} B_r R l d\phi$$

where B_r is the radial flux through the coil. And, if B_r is sinusoidally distributed this will have a peak value of

$$\phi_p = \frac{2RlB_r}{p}$$

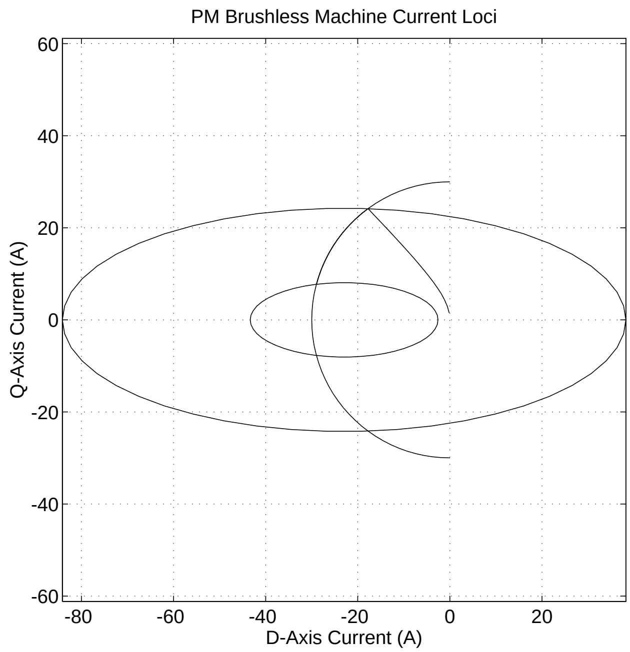

Figure 8: Operating Current Loci of Example Machine

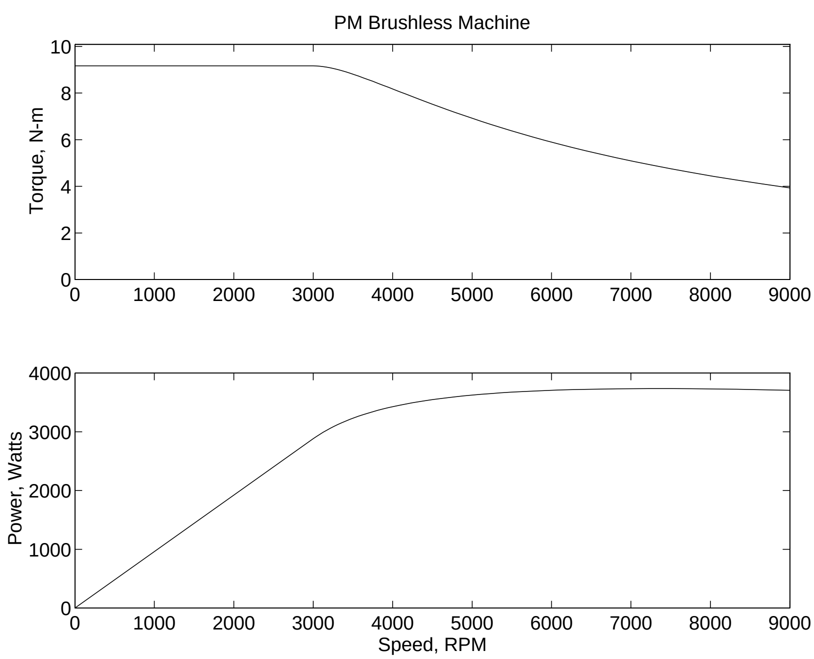

Figure 9: Torque- and Power-Speed Capability

Now, if the actual winding has N_a turns, and using the pitch and breadth factors derived in Appendix 1, the total flux linked is simply:

$$\lambda_f = \frac{2RlB_1N_ak_w}{p} \tag{30}$$

where

$$\begin{aligned} k_w &= k_pk_b \\ k_p &= \sin\frac{\alpha}{2} \\ k_b &= \frac{\sin m\frac{\gamma}{2}}{m\sin\frac{\gamma}{2}} \end{aligned}$$

The angle α is the *pitch* angle,

$$\alpha = 2\pi p\frac{N_p}{N_s}$$

where N_p is the coil span (in slots) and N_s is the total number of slots in the stator. The angle γ is the slot electrical angle:

$$\gamma = \frac{2\pi p}{N_s}$$

Now, what remains to be found is the space fundamental magnetic flux density B_1. In Appendix 2 it is shown that, for magnets in a surface-mount geometry, the magnetic field at the surface of the magnetic gap is:

$$B_1 = \mu_0 M_1 k_g \tag{31}$$

where the space-fundamental magnetization is:

$$M_1 = \frac{B_r}{\mu_0}\frac{4}{\pi}\sin\frac{p\theta_m}{2}$$

where B_r is remanent flux density of the permanent magnets and θ_m is the magnet angle.

and where the factor that describes the geometry of the magnetic gap depends on the case. For magnets inside and $p \neq 1$,

$$k_g = \frac{R_s^{p-1}}{R_s^{2p} - R_i^{2p}}\left(\frac{p}{p+1}\left(R_2^{p+1} - R_1^{p+1}\right) + \frac{p}{p-1}R_i^{2p}\left(R_1^{1-p} - R_2^{1-p}\right)\right)$$

For magnets inside and $p = 1$,

$$k_g = \frac{1}{R_s^2 - R_i^2}\left(\frac{1}{2}\left(R_2^2 - R_1^2\right) + R_i^2\log\frac{R_2}{R_1}\right)$$

For the case of magnets outside and $p \neq 1$:

$$k_g = \frac{R_i^{p-1}}{R_s^{2p} - R_i^{2p}}\left(\frac{p}{p+1}\left(R_2^{p+1} - R_1^{p+1}\right) + \frac{p}{p-1}R_s^{2p}\left(R_1^{1-p} - R_2^{1-p}\right)\right)$$

and for magnets outside and $p = 1$,

$$k_g = \frac{1}{R_s^2 - R_i^2}\left(\frac{1}{2}\left(R_2^2 - R_1^2\right) + R_s^2 \log \frac{R_2}{R_1}\right)$$

Where R_s and R_i are the outer and inner magnetic boundaries, respectively, and R_2 and R_1 are the outer and inner boundaries of the magnets.

Note that for the case of a small gap, in which both the *physical* gap g and the magnet thickness h_m are both much less than rotor radius, it is straightforward to show that all of the above expressions approach what one would calculate using a simple, one-dimensional model for the permanent magnet:

$$k_g \rightarrow \frac{h_m}{g + h_m}$$

This is the whole story for the winding-in-slot, narrow air-gap, surface magnet machine. For air-gap armature windings, it is necessary to take into account the radial dependence of the magnetic field.

4.2 Air-Gap Armature Windings

With no windings in slots, the conventional definition of winding factor becomes difficult to apply. If, however, each of the phase belts of the winding occupies an angular extent θ_w, then the equivalent to (31) is:

$$k_w = \frac{\sin p \frac{\theta_w}{2}}{p \frac{\theta_w}{2}}$$

Next, assume that the "density" of conductors within each of the phase belts of the armature winding is uniform, so that the density of turns as a function of radius is:

$$N(r) = \frac{2 N_a r}{R_{wo}^2 - R_{wi}^2}$$

This just expresses the fact that there is more azimuthal room at larger radii, so with uniform density the number of turns as a function of radius is linearly dependent on radius. Here, R_{wo} and R_{wi} are the outer and inner radii, respectively, of the winding.

Now it is possible to compute the flux linked due to a magnetic field distribution:

$$\lambda_f = \int_{R_{wi}}^{R_{wo}} \frac{2 l N_a k_w r}{p} \frac{2r}{R_{wo}^2 - R_{wi}^2} \mu_0 H_r(r) dr \tag{32}$$

Note the form of the magnetic field as a function of radius expressed in 80 and 81 of the second appendix. For the "winding outside" case it is:

$$H_r = A\left(r^{p-1} + R_s^{2p} r^{-p-1}\right)$$

Then a winding with all its turns concentrated at the outer radius $r = R_{wo}$ would link flux:

$$\lambda_c = \frac{2 l R_{wo} k_w}{p} \mu_0 H_r(R_{wo}) = \frac{2 l R_{wo} k_w}{p} \mu_0 A\left(R_{wo}^{p-1} + R_s^{2p} R_{wo}^{-p-1}\right)$$

Carrying out (32), it is possible, then, to express the flux linked by a thick winding to the flux that would have been linked by a radially concentrated winding at its outer surface by:

$$k_t = \frac{\lambda_f}{\lambda_c}$$

where, for the winding outside, $p \neq 2$ case:

$$k_t = \frac{2}{(1-x^2)(1+\xi^{2p})}\left(\frac{(1-x^{2+p})\,\xi^{2p}}{2+p} + \frac{1-x^{2-p}}{2-p}\right) \tag{33}$$

where we have used the definitions $\xi = R_{wo}/R_s$ and $x = R_{wi}/R_{wo}$. In the case of winding outside, $p = 2$,

$$k_t = \frac{2}{(1-x^2)(1+\xi^{2p})}\left(\frac{(1-x^4)\,\xi^4}{4} - \log x\right) \tag{34}$$

In a very similar way, we can define a winding factor for a thick winding in which the reference radius is at the inner surface. (Note: this is done because the inner surface of the inside winding is likely to be coincident with the inner ferromagnetic surface, as the outer surface of the outer winding ls likely to be coincident with the outer ferromagnetic surface). For $p \neq 2$:

$$k_t = \frac{2x^{-p}}{(1-x^2)(1+\eta^{2p})}\left(\frac{1-x^{2+p}}{2+p} + (\eta x)^{2p}\frac{1-x^{2-p}}{2-p}\right) \tag{35}$$

and for $p = 2$:

$$k_t = \frac{2x^{-2}}{(1-x^2)(1+\eta^{2p})}\left(\frac{1-x^4}{4} - (\eta x)^4 \log x\right) \tag{36}$$

where $\eta = R_i/R_{wi}$

So, in summary, the flux linked by an air-gap armature is given by:

$$\lambda_f = \frac{2RlB_1N_ak_wk_t}{p} \tag{37}$$

where B_1 is the flux density at the outer radius of the physical winding (for outside winding machines) or at the inner radius of the physical winding (for inside winding machines). Note that the additional factor k_t is a bit more than one (it approaches unity for thin windings), so that, for small pole numbers and windings that are not too thick, it is almost correct and in any case "conservative" to take it to be one.

4.3 Interior Magnet Motors:

For the flux concentrating machine, it is possible to estimate air-gap flux density using a simple reluctance model.

The air- gap permeance of one pole piece is:

$$\wp_{ag} = \mu_0 l \frac{R\theta_p}{g}$$

where θ_p is the angular width of the pole piece.

And the incremental permeance of a magnet is:

$$\wp_m = \mu_0 \frac{h_m l}{w_m}$$

The magnet sees a *unit permeance* consisting of its own permeance in series with one half of each of two pole pieces (in series) :

$$\wp_u = \frac{\wp_{ag}}{\wp_m} = \frac{R\theta_p}{4g}\frac{w_m}{h_m}$$

Magnetic flux density in the *magnet* is:

$$B_m = B_0 \frac{\wp_u}{1+\wp_u}$$

And then flux density in the *air gap* is:

$$B_g = \frac{2h_m}{R\theta_p} B_m = B_0 \frac{2h_m w_m}{4gh_m + R\theta_p w_m}$$

The space fundamental of that can be written as:

$$B_1 = \frac{4}{\pi} \sin \frac{p\theta_p}{2} B_0 \frac{w_m}{2g} \gamma_m$$

where we have introduced the shorthand:

$$\gamma_m = \frac{1}{1 + \frac{w_m}{g}\frac{\theta_p}{4}\frac{R}{h_m}}$$

The flux linkage is then computed as before:

$$\lambda_f = \frac{2RlB_1 N_a k_w}{p} \tag{38}$$

4.4 Winding Inductances

The next important set of parameters to compute are the d- and q- axis inductances of the machine. We will consider three separate cases, the winding-in-slot, surface magnet case, which is magnetically "round", or non-salient, the air-gap winding case, and the flux concentrating case which is salient, or has different direct- and quadrature- axis inductances.

4.4.1 Surface Magnets, Windings in Slots

In this configuration there is no saliency, so that $L_d = L_q$. There are two principal parts to inductance, the air-gap inductance and slot leakage inductance. Other components, including end turn leakage, may be important in some configurations, and they would be computed in the same way as for an induction machine. If magnet thickness is not too great, we may make the narrow air-gap assumption, in which case the fundamental part of air-gap inductance is:

$$L_{d1} = \frac{q}{2}\frac{4}{\pi}\frac{\mu_0 N_a^2 k_w^2 l R_s}{p^2(g+h_m)} \tag{39}$$

Here, g is the magnetic gap, including the physical rotational gap and any magnet retaining means that might be used. h_m is the magnet thickness.

Since the magnet thickness is included in the air-gap, the air-gap permeance may not be very large, so that slot leakage inductance may be important. To estimate this, assume that the slot shape is rectangular, characterized by the following dimensions:

h_s height of the main portion of the slot
w_s width of the top of the main portion of the slot
h_d height of the slot depression
w_d slot depression opening

Of course not all slots are rectangular: in fact in most machines the slots are trapezoidal in shape to maintain teeth cross-sections that are radially uniform. However, only a very small error (a few percent) is incurred in calculating slot permeance if the slot is assumed to be rectangular and the *top* width is used (that is the width closest to the air-gap). Then the slot permeance is, per unit length:

$$\mathcal{P} = \mu_0 \left(\frac{1}{3} \frac{h_s}{w_s} + \frac{h_d}{w_d} \right)$$

Assume for the rest of this discussion a standard winding, with m slots in each phase belt (this assumes, then, that the total number of slots is $N_s = 2pqm$), and each slot holds two half-coils. (A half-coil is one side of a coil which, of course, is wound in two slots). If each coil has N_c turns (meaning $N_a = 2pmN_c$) , then the contribution to phase self-inductance of *one* slot is, if both half-coils are from the same phase, $4l\mathcal{P}N_c^2$. If the half-coils are from different phases, then the contribution to self inductance is $l\mathcal{P}N_c^2$ and the magnitude of the contribution to mutual inductance is $l\mathcal{P}N_c^2$. (Some caution is required here. For three phase windings the mutual inductance is negative, so are the senses of the currents in the two other phases, so the impact of "mutual leakage" is to increase the reactance. This will be true for other numbers of phases as well, even if the algebraic sign of the mutual leakage inductance is positive, in which case so will be the sense of the other- phase current.)

We will make two other assumptions here. The standard one is that the winding "coil throw", or span between sides of a coil, is $\frac{N_s}{2p} - N_{sp}$. N_{sp} is the coil "short pitch". The other is that each phase belt will overlap with, at most two other phases: the ones on either side in sequence. This last assumption is immediately true for three- phase windings (because there *are* only two other phases. It is also likely to be true for any reasonable number of phases.

Noting that each phase occupies $2p(m - N_{sp})$ slots with both coil halves in the same slot and $2pN_{sp}$ slots in which one coil half shares a slot with each of two different phases, we can write down the two components of slot leakage inductance, self- and mutual:

$$\begin{aligned} L_{as} &= 2pl \left[(m - N_{sp}) (2N_c)^2 + 2N_{sp}N_c^2 \right] \\ L_{am} &= -2plN_{sp}N_c^2 \end{aligned}$$

For a three- phase machine, then, the total slot leakage inductance is:

$$L_a = L_{as} - L_{am} = 2pl\mathcal{P}N_c^2 (4m - N_{sp})$$

For a uniform, symmetric winding with an odd number of phases, it is possible to show that the effective slot leakage inductance is:

$$L_a = L_{as} - 2L_{am} \cos \frac{2\pi}{q}$$

Total synchronous inductance is the sum of air-gap and leakage components: so far this is:

$$L_d = L_{d1} + L_a$$

4.4.2 Air-Gap Armature Windings

It is shown in Appendix 1 that the inductance of a single-phase of an air-gap winding is:

$$L_a = \sum_n L_{np}$$

where the harmonic components are:

$$\begin{aligned} L_k &= \frac{8}{\pi} \frac{\mu_0 l k_{wn}^2 N_a^2}{k(1-x^2)^2} \left[\frac{\left(1 - x^{2-k}\gamma^{2k}\right)\left(1 - x^{2+k}\right)}{(4-k^2)(1-\gamma^{2k})} \right. \\ &+ \frac{\xi^{2k}\left(1 - x^{k+2}\right)^2}{(2+k)^2(1-\gamma^{2k})} + \frac{\xi^{-2k}\left(1-x^{2-k}\right)^2}{(2-k)^2(\gamma^{-2k}-1)} \\ &+ \left. \frac{\left(1-\gamma^{-2k}x^{2+k}\right)\left(1-x^{2-k}\right)}{(4-k^2)(\gamma^{-2k}-1)} - \frac{k}{4-k^2}\frac{1-x^2}{2} \right] \end{aligned}$$

where we have used the following shorthand coefficients:

$$\begin{aligned} x &= \frac{R_{wi}}{R_{wo}} \\ \gamma &= \frac{R_i}{R_s} \\ \xi &= \frac{R_{wo}}{R_s} \end{aligned}$$

This fits into the conventional inductance framework:

$$L_n = \frac{4}{\pi} \frac{\mu_0 N_a^2 R_s L k_{wn}^2}{N^2 p^2 g} k_a$$

if we assign the "thick armature" coefficient to be:

$$\begin{aligned} k_a &= \frac{2gk}{R_{wo}} \frac{1}{(1-x^2)^2} \left[\frac{\left(1 - x^{2-k}\gamma^{2k}\right)\left(1 - x^{2+k}\right)}{(4-k^2)(1-\gamma^{2k})} \right. \\ &+ \frac{\xi^{2k}\left(1 - x^{k+2}\right)^2}{(2+k)^2(1-\gamma^{2k})} + \frac{\xi^{-2k}\left(1-x^{2-k}\right)^2}{(2-k)^2(\gamma^{-2k}-1)} \\ &+ \left. \frac{\left(1-\gamma^{-2k}x^{2+k}\right)\left(1-x^{2-k}\right)}{(4-k^2)(\gamma^{-2k}-1)} - \frac{k}{4-k^2}\frac{1-x^2}{2} \right] \end{aligned}$$

and $k = np$ and $g = R_s - R_i$ is the conventionally defined "air gap". If the aspect ratio R_i/R_s is not too far from unity, neither is k_a. In the case of $p = 2$, the fundamental component of k_a is:

$$k_a = \frac{2gk}{R_{wo}} \frac{1}{(1-x^2)^2} \left[\frac{1-x^4}{8} - \frac{2\gamma^4 + x^4(1-\gamma^4)}{4(1-\gamma^4)} \log x + \frac{\gamma^4}{\xi^4(1-\gamma^4)} (\log x)^2 + \frac{\xi^4 (1-x^4)^2}{16(1-\gamma^4)} \right]$$

For a q-phase winding, a good approximation to the inductance is given by just the first space harmonic term, or:

$$L_d = \frac{q}{2} \frac{4}{\pi} \frac{\mu_0 N_a^2 R_s L k_{wn}^2}{n^2 p^2 g} k_a$$

4.4.3 Internal Magnet Motor

The permanent magnets will have an effect on reactance because the magnets are in the main flux path of the armature. Further, they affect direct and quadrature reactances differently, so that the machine will be salient. Actually, the effect on the direct axis will likely be greater, so that this type of machine will exhibit "negative" saliency: the quadrature axis reactance will be larger than the direct- axis reactance.

A full- pitch coil aligned with the direct axis of the machine would produce flux density:

$$B_r = \frac{\mu_0 N_a I}{2g\left(1 + \frac{R\theta_p}{4g}\frac{w_m}{h_m}\right)}$$

Note that only the pole area is carrying useful flux, so that the space fundamental of radial flux density is:

$$B_1 = \frac{\mu_0 N_a I}{2g} \frac{4}{\pi} \frac{\sin\frac{p\theta_m}{2}}{1 + \frac{w_m}{h_m}\frac{R\theta_p}{4g}}$$

Then, since the flux linked by the winding is:

$$\lambda_a = \frac{2RlN_a k_w B_1}{p}$$

The d- axis inductance, including mutual phase coupling, is (for a q- phase machine):

$$L_d = \frac{q}{2} \frac{4}{\pi} \frac{\mu_0 N_a^2 R l k_w^2}{p^2 g} \gamma_m \sin\frac{p\theta_p}{2}$$

The quadrature axis is quite different. On that axis, the armature does *not* tend to push flux through the magnets, so they have only a minor effect. What effect they *do* have is due to the fact that the magnets produce a space in the active air- gap. Thus, while a full- pitch coil aligned with the quadrature axis will produce an air- gap flux density:

$$B_r = \frac{\mu_0 N I}{g}$$

the space fundamental of that will be:

$$B_1 = \frac{\mu_0 N I}{g} \frac{4}{\pi} \left(1 - \sin \frac{p\theta_t}{2} \right)$$

where θ_t is the angular width taken out of the pole by the magnets.

So that the expression for quadrature axis inductance is:

$$L_q = \frac{q}{2} \frac{4}{\pi} \frac{\mu_0 N_a^2 R l k_w^2}{p^2 g} \left(1 - \sin \frac{p\theta_t}{2} \right)$$

5 Current Rating and Resistance

The last part of machine rating is its current capability. This is heavily influenced by cooling methods, for the principal limit on current is the heating produced by resistive dissipation. Generally, it is possible to do first-order design estimates by assuming a current density that can be handled by a particular cooling scheme. Then, in an air-gap winding:

$$N_a I_a = \left(R_{wo}^2 - R_{wi}^2 \right) \frac{\theta_{we}}{2} J_a$$

and note that, usually, the armature fills the azimuthal space in the machine:

$$2q\theta_{we} = 2\pi$$

For a winding in slots, nearly the same thing is true: if the rectangular slot model holds true:

$$2qN_a I_a = N_s h_s w_s J_s$$

where we are using J_s to note *slot* current density. Now, suppose we can characterize the total slot area by a "space factor" λ_s which is the ratio between total slot area and the annulus occupied by the slots: for the rectangular slot model:

$$\lambda_s = \frac{N_s h_s w_s}{\pi \left(R_{wo}^2 - R_{wi}^2 \right)}$$

where $R_{wi} = R + h_d$ and $R_{wo} = R_{wi} + h_s$ in a normal, stator outside winding. In this case, $J_a = J_s \lambda_s$ and the two types of machines can be evaluated in the same way.

It would seem apparent that one would want to make λ_s as large as possible, to permit high currents. The limit on this is that the magnetic teeth between the conductors must be able to carry the air-gap flux, and making them too narrow would cause them to saturate. The peak of the time fundamental magnetic field in the teeth is, for example,

$$B_t = B_1 \frac{2\pi R}{N_s w_t}$$

where w_t is the width of a stator tooth:

$$w_t = \frac{2\pi(R + h_d)}{N_s} - w_s$$

so that

$$B_t \approx \frac{B_1}{1 - \lambda_s}$$

5.1 Resistance

Winding resistance may be estimated as the length of the stator conductor divided by its area and its conductivity. The length of the stator conductor is:

$$l_c = 2lN_a f_e$$

where the "end winding factor" f_e is used to take into account the extra length of the end turns (which is usually *not* negligible). The *area* of each turn of wire is, for an air-gap winding :

$$A_w = \frac{\theta_{we}}{2} \frac{R_{wo}^2 - R_{wi}^2}{N_a} \lambda_w$$

where λ_w, the "packing factor" relates the area of conductor to the total area of the winding. The resistance is then just:

$$R_a = \frac{4lN_a^2}{\theta_{we}\left(R_{wo}^2 - R_{wi}^2\right)\lambda_w \sigma}$$

and, of course, σ is the conductivity of the conductor.

For windings in slots the expression is almost the same, simply substituting the total slot area:

$$R_a = \frac{2qlN_a^2}{N_s h_s w_s \lambda_w \sigma}$$

The end turn allowance depends strongly on how the machine is made. One way of estimating what it might be is to assume that the end turns follow a roughly circular path from one side of the machine to the other. The radius of this circle would be, very roughly, R_w/p, where R_w is the average radius of the winding: $R_w \approx (R_{wo} + R_{wi})/2$

Then the end-turn allowance would be:

$$f_e = 1 + \frac{\pi R_w}{pl}$$

6 Appendix 1: Air-Gap Winding Inductance

In this appendix we use a simple two-dimensional model to estimate the magnetic fields and then inductances of an air-gap winding. The principal limiting assumption here is that the winding is uniform in the $\hat{z}$ direction, which means it is long in comparison with its radii. This is generally not true, nevertheless the answers we will get are not too far from being correct. The *style* of analysis used here can be carried into a three-dimensional, or quasi-three dimensional domain to get much more precise answers, at the expense of a very substantial increase in complexity.

The coordinate system to be used is shown in Figure 10. To maintain generality we have four radii: R_i and R_s are ferromagnetic boundaries, and would of course correspond with the machine shaft and the stator core. The winding itself is carried between radii R_1 and R_2, which correspond with radii R_{wi} and R_{wo} in the body of the text. It is assumed that the armature is carrying a current in the z- direction, and that this current is uniform in the radial dimension of the armature. If a single phase of the armature is carrying current, that current will be:

$$J_{z0} = \frac{N_a I_a}{\frac{\theta_{we}}{2}\left(R_2^2 - R_1^2\right)}$$

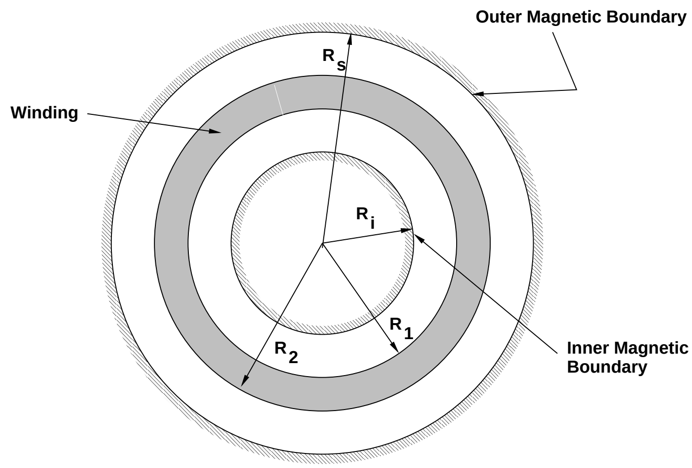

Figure 10: Coordinate System for Inductance Calculation

over the annular wedge occupied by the phase. The resulting distribution can be fourier analyzed, and the n-th harmonic component of this will be (assuming the coordinate system has been chosen appropriately):

$$J_{zn} = \frac{4}{n\pi} J_{z0} \sin n \frac{\theta_{we}}{2} = \frac{4}{\pi} \frac{N_a I_a}{R_2^2 - R_1^2} k_{wn}$$

where the n-th harmonic winding factor is:

$$k_{wn} = \frac{\sin n \frac{\theta_{we}}{2}}{n \frac{\theta_{we}}{2}}$$

and note that θ_{we} is the *electrical* winding angle:

$$\theta_{we} = p\theta_w$$

Now, it is easiest to approach this problem using a vector potential. Since the divergence of flux density is zero, it is possible to let the magnetic flux density be represented by the curl of a vector potential:

$$\overline{B} = \nabla \times \overline{A}$$

Taking the curl of *that*:

$$\nabla \times \left(\nabla \times \overline{A} \right) = \mu_0 \overline{J} = \nabla \nabla \cdot \overline{A} - \nabla^2 \overline{A}$$

and using the Coulomb gage

$$\nabla \cdot \overline{A} = 0$$

we have a reasonable tractable partial differential equation in the vector potential:

$$\nabla^2 \overline{A} = -\mu_0 \overline{J}$$

Now, since in our assumption there is only a z- directed component of $\overline{J}$, we can use that one component, and in circular cylindrical coordinates that is:

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial A_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} A_z = -\mu_0 J_z$$

For this problem, all variables will be varying sinusoidally with angle, so we will assume that angular dependence $e^{jk\theta}$. Thus:

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial A_z}{\partial r} - \frac{k^2}{r^2}A_z = -\mu_0 J_z \tag{40}$$

This is a three-region problem. Note the regions as:

i	$R_i < r < R_1$
w	$R_1 < r < R_2$
o	$R_2 < r < R_s$

For i and o, the current density is zero and an appropriate solution to (40) is:

$$A_z = A_+ r^k + A_- r^{-k}$$

In the region of the winding, w, a particular solution must be used in addition to the homogeneous solution, and

$$A_z = A_+ r^k + A_- r^{-k} + A_p$$

where, for $k \neq 2$,

$$A_p = -\frac{\mu_0 J_z r^2}{4 - k^2}$$

or, if $k = 2$,

$$A_p = -\frac{\mu_0 J_z r^2}{4}\left(\log r - \frac{1}{4}\right)$$

And, of course, the two pertinent components of the magnetic flux density are:

$$\begin{aligned} B_r &= \frac{1}{r}\frac{\partial A_z}{\partial \theta} \\ B_\theta &= -\frac{\partial A_z}{\partial r} \end{aligned}$$

Next, it is necessary to match boundary conditions. There are six free variables and correspondingly there must be six of these boundary conditions. They are the following:

- At the inner and outer magnetic boundaries, $r = R_i$ and $r = R_s$, the azimuthal magnetic field must vanish.
- At the inner and outer radii of the winding itself, $r = R_1$ and $r = R_2$, *both* radial and azimuthal magnetic field must be continuous.

These conditions may be summarized by:

$$\begin{aligned} kA_+^i R_i^{k-1} - kA_-^i R_i^{-k-1} &= 0 \\ kA_+^o R_s^{k-1} - kA_-^o R_s^{-k-1} &= 0 \\ A_+^w R_2^{k-1} + A_-^w R_2^{-k-1} - \frac{\mu_0 J_z R_2}{4 - k^2} &= A_+^o R_2^{k-1} + A_-^o R_2^{-k-1} \end{aligned}$$

$$
\begin{aligned}
-kA_+^w R_2^{k-1} + kA_-^w R_2^{-k-1} + \frac{2\mu_0 J_z R_2}{4-k^2} &= -kA_+^o R_2^{k-1} + kA_-^o R_2^{-k-1} \\
A_+^w R_1^{k-1} + A_-^w R_1^{-k-1} - \frac{\mu_0 J_z R_1}{4-k^2} &= A_+^i R_1^{k-1} + A_-^i R_1^{-k-1} \\
-kA_+^w R_1^{k-1} + kA_-^w R_1^{-k-1} + \frac{2\mu_0 J_z R_1}{4-k^2} &= -kA_+^i R_1^{k-1} + kA_-^i R_1^{-k-1}
\end{aligned}
$$

Note that we are carrying this out here only for the case of $k \neq 2$. The $k = 2$ case may be obtained by substituting its particular solution in at the beginning or by using L'Hopital's rule on the final solution. This set may be solved (it is a bit tedious but quite straightforward) to yield, for the winding region:

$$
\begin{aligned}
A_z = \frac{\mu_0 J_z}{2k} & \left[\left(\frac{R_s^{2k} R_2^{2-k} - R_i^{2k} R_1^{2-k}}{(2-k)\left(R_s^{2k} - R_i^{2k}\right)} + \frac{R_2^{2+k} - R_1^{2+k}}{(2+k)\left(R_s^{2k} - R_i^{2k}\right)} \right) r^k \right. \\
& \left. + \left(\frac{R_2^{2-k} - R_1^{2-k}}{(2-k)\left(R_i^{-2k} - R_s^{-2k}\right)} + \frac{R_s^{-2k} R_2^{2+k} - R_i^{-2k} R_1^{2+k}}{(2+k)\left(R_i^{-2k} - R_s^{-2k}\right)} \right) r^{-k} - \frac{2k}{4-k^2} r^2 \right]
\end{aligned}
$$

Now, the inductance linked by any single, full-pitched loop of wire located with one side at azimuthal position θ and radius r is:

$$
\lambda_i = 2lA_z(r, \theta)
$$

To extend this to the whole winding, we integrate over the area of the winding the incremental flux linked by each element times the turns density. This is, for the n-th harmonic of flux linked:

$$
\lambda_n = \frac{4lk_{wn}N_a}{R_2^2 - R_1^2} \int_{R_1}^{R_2} A_z(r) r dr
$$

Making the appropriate substitutions for current into the expression for vector potential, this becomes:

$$
\begin{aligned}
\lambda_n = \frac{8}{\pi} \frac{\mu_0 l k_{wn}^2 N_a^2 I_a}{k\left(R_2^2 - R_1^2\right)^2} & \left[\left(\frac{R_s^{2k} R_2^{2-k} - R_i^{2k} R_1^{2-k}}{(2-k)\left(R_s^{2k} - R_i^{2k}\right)} + \frac{R_2^{2+k} - R_1^{2+k}}{(2+k)\left(R_2^{2k} - R_i^{2k}\right)} \right) \frac{R_2^{k+2} - R_1^{k+2}}{k+2} \right. \\
& \left. + \left(\frac{R_2^{2-k} - R_1^{2-k}}{(2-k)\left(R_i^{-2k} - R_s^{-2k}\right)} + \frac{R_s^{-2k} R_2^{2+k} - R_i^{-2k} R_1^{2+k}}{(2+k)\left(R_i^{-2k} - R_s^{-2k}\right)} \right) \frac{R_2^{2-k} - R_1^{2-k}}{2-k} - \frac{2k}{4-k^2} \frac{R_2^4 - R_1^4}{4} \right]
\end{aligned}
$$

7 Appendix 2: Permanent Magnet Field Analysis

This section is a field analysis of the kind of radially magnetized, permanent magnet structures commonly used in electric machinery. It is a fairly general analysis, which will be suitable for use with either surface or in-slot windings, and for the magnet inside or the magnet outside case.

This is a two-dimensional layout suitable for situations in which field variation along the length of the structure is negligible.

8 Layout

The assumed geometry is shown in Figure 11. Assumed iron (highly permeable) boundaries are at radii R_i and R_s. The permanent magnets, assumed to be polarized radially and alternately (i.e. North-South ...), are located between radii R_1 and R_2. We assume there are p pole pairs ($2p$ magnets) and that each magnet subsumes an electrical angle of θ_{me}. The electrical angle is just p times the physical angle, so that if the magnet angle were $\theta_{me} = \pi$, the magnets would be touching.

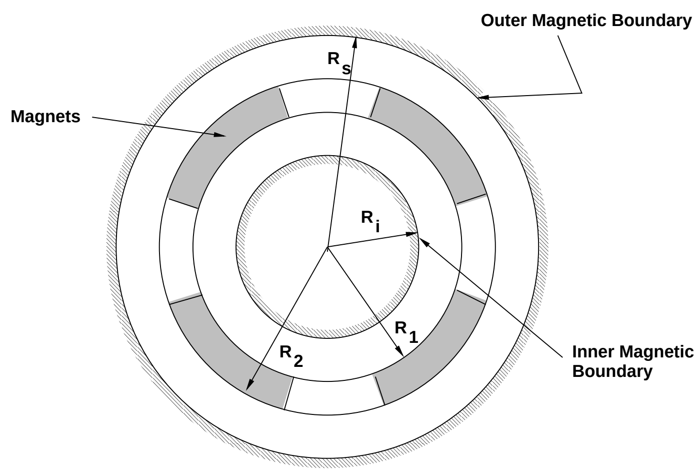

Figure 11: Axial View of Magnetic Field Problem

If the magnets are arranged so that the radially polarized magnets are located around the azimuthal origin ($\theta = 0$), the space fundamental of magnetization is:

$$\overline{M} = \overline{i}_r M_0 \cos p\theta \tag{41}$$

where the fundamental magnitude is:

$$M_0 = \frac{4}{\pi} \sin \frac{\theta_{me}}{2} \frac{B_{\text{rem}}}{\mu_0} \tag{42}$$

and B_{rem} is the remanent magnetization of the permanent magnet.

Since there is no current anywhere in this problem, it is convenient to treat magnetic field as the gradient of a scalar potential:

$$\overline{H} = -\nabla\psi \tag{43}$$

The divergence of this is:

$$\nabla^2\psi = -\nabla \cdot \overline{H} \tag{44}$$

Since magnetic *flux* density is divergence-free,

$$\nabla \cdot \overline{B} = 0 \tag{45}$$

we have:

$$\nabla \cdot \overline{H} = -\nabla \cdot \overline{M} \tag{46}$$

or:

$$\nabla^2 \psi = \nabla \cdot \overline{M} = \frac{1}{r} M_0 \cos p\theta \tag{47}$$

Now, if we let the magnetic scalar potential be the sum of *particular* and *homogeneous* parts:

$$\psi = \psi_p + \psi_h \tag{48}$$

where $\nabla^2 \psi_h = 0$, then:

$$\nabla^2 \psi_p = \frac{1}{r} M_0 \cos p\theta \tag{49}$$

We can find a suitable solution to the *particular* part of this in the region of magnetization by trying:

$$\psi_p = C r^\gamma \cos p\theta \tag{50}$$

Carrying out the Laplacian on this:

$$\nabla^2 \psi_p = C r^{\gamma - 2} \left(\gamma^2 - p^2 \right) \cos p\theta = \frac{1}{r} M_0 \cos p\theta \tag{51}$$

which works if $\gamma = 1$, in which case:

$$\psi_p = \frac{M_0 r}{1 - p^2} \cos p\theta \tag{52}$$

Of course this solution holds only for the region of the magnets: $R_1 < r < R_2$, and is zero for the regions outside of the magnets.

A suitable *homogeneous* solution satisfies Laplace's equation, $\nabla^2 \psi_h = 0$, and is in general of the form:

$$\psi_h = A r^p \cos p\theta + B r^{-p} \cos p\theta \tag{53}$$

Then we may write a trial *total* solution for the flux density as:

$$R_i < r < R_1 \quad \psi = \left(A_1 r^p + B_1 r^{-p} \right) \cos p\theta \tag{54}$$

$$R_1 < r < R_2 \quad \psi = \left(A_2 r^p + B_2 r^{-p} + \frac{M_0 r}{1 - p^2} \right) \cos p\theta \tag{55}$$

$$R_2 < r < R_s \quad \psi = \left(A_3 r^p + B_3 r^{-p} \right) \cos p\theta \tag{56}$$

The boundary conditions at the inner and outer (assumed infinitely permeable) boundaries at $r = R_i$ and $r = R_s$ require that the azimuthal field vanish, or $\frac{\partial \psi}{\partial \theta} = 0$, leading to:

$$B_1 = -R_i^{2p} A_1 \tag{57}$$

$$B_3 = -R_s^{2p} A_3 \tag{58}$$

At the magnet inner and outer radii, H_θ and B_r must be continuous. These are:

$$H_\theta = -\frac{1}{r}\frac{\partial\psi}{\partial\theta} \quad (59)$$

$$B_r = \mu_0\left(-\frac{\partial\psi}{\partial r} + M_r\right) \quad (60)$$

These become, at $r = R_1$:

$$-pA_1\left(R_1^{p-1} - R_i^{2p}R_1^{-p-1}\right) = -p\left(A_2R_1^{p-1} + B_2R_1^{-p-1}\right) - p\frac{M_0}{1-p^2} \quad (61)$$

$$-pA_1\left(R_1^{p-1} + R_i^{2p}R_1^{-p-1}\right) = -p\left(A_2R_1^{p-1} - B_2R_1^{-p-1}\right) - \frac{M_0}{1-p^2} + M_0 \quad (62)$$

and at $r = R_2$:

$$-pA_3\left(R_2^{p-1} - R_s^{2p}R_2^{-p-1}\right) = -p\left(A_2R_2^{p-1} + B_2R_2^{-p-1}\right) - p\frac{M_0}{1-p^2} \quad (63)$$

$$-pA_3\left(R_2^{p-1} + R_s^{2p}R_2^{-p-1}\right) = -p\left(A_2R_2^{p-1} - B_2R_2^{-p-1}\right) - \frac{M_0}{1-p^2} + M_0 \quad (64)$$

Some small-time manipulation of these yields:

$$A_1\left(R_1^p - R_i^{2p}R_1^{-p}\right) = A_2R_1^p + B_2R_1^{-p} + R_1\frac{M_0}{1-p^2} \quad (65)$$

$$A_1\left(R_1^p + R_i^{2p}R_1^{-p}\right) = A_2R_1^p - B_2R_1^{-p} + pR_1\frac{M_0}{1-p^2} \quad (66)$$

$$A_3\left(R_2^p - R_s^{2p}R_2^{-p}\right) = A_2R_2^p + B_2R_2^{-p} + R_2\frac{M_0}{1-p^2} \quad (67)$$

$$A_3\left(R_2^p + R_s^{2p}R_2^{-p}\right) = A_2R_2^p - B_2R_2^{-p} + pR_2\frac{M_0}{1-p^2} \quad (68)$$

Taking sums and differences of the first and second and then third and fourth of these we obtain:

$$2A_1R_1^p = 2A_2R_1^p + R_1M_0\frac{1+p}{1-p^2} \quad (69)$$

$$2A_1R_i^{2p}R_1^{-p} = -2B_2R_1^{-p} + R_1M_0\frac{p-1}{1-p^2} \quad (70)$$

$$2A_3R_2^p = 2A_2R_2^p + R_2M_0\frac{1+p}{1-p^2} \quad (71)$$

$$2A_3R_s^{2p}R_2^{-p} = -2B_2R_2^{-p} + R_2M_0\frac{p-1}{1-p^2} \quad (72)$$

and then multiplying through by appropriate factors (R_2^p and R_1^p) and then taking sums and differences of *these*,

$$\left(A_1 - A_3\right)R_1^pR_2^p = \left(R_1R_2^p - R_2R_1^p\right)\frac{M_0}{2}\frac{p+1}{1-p^2} \quad (73)$$

$$\left(A_1R_i^{2p} - A_3R_s^{2p}\right)R_1^{-p}R_2^{-p} = \left(R_1R_2^{-p} - R_2R_1^{-p}\right)\frac{M_0}{2}\frac{p-1}{1-p^2} \quad (74)$$

Dividing through by the appropriate groups:

$$A_1 - A_3 = \frac{R_1 R_2^p - R_2 R_1^p}{R_1^p R_2^p} \frac{M_0}{2} \frac{1+p}{1-p^2} \tag{75}$$

$$A_1 R_i^{2p} - A_3 R_s^{2p} = \frac{R_1 R_2^{-p} - R_2 R_1^{-p}}{R_1^{-p} R_2^{-p}} \frac{M_0}{2} \frac{p-1}{1-p^2} \tag{76}$$

and then, by multiplying the top equation by R_s^{2p} and subtracting:

$$A_1 \left(R_s^{2p} - R_i^{2p}\right) = \left(\frac{R_1 R_2^p - R_2 R_1^p}{R_1^p R_2^p} \frac{M_0}{2} \frac{1+p}{1-p^2}\right) R_s^{2p} - \frac{R_1 R_2^{-p} - R_2 R_1^{-p}}{R_1^{-p} R_2^{-p}} \frac{M_0}{2} \frac{p-1}{1-p^2} \tag{77}$$

This is readily solved for the field coefficients A_1 and A_3:

$$A_1 = -\frac{M_0}{2\left(R_s^{2p} - R_i^{2p}\right)} \left(\frac{p+1}{p^2-1}\left(R_1^{1-p} - R_2^{1-p}\right) R_s^{2p} + \frac{p-1}{p^2-1}\left(R_2^{1+p} - R_1^{1+p}\right)\right) \tag{78}$$

$$A_3 = -\frac{M_0}{2\left(R_s^{2p} - R_i^{2p}\right)} \left(\frac{1}{1-p}\left(R_1^{1-p} - R_2^{1-p}\right) R_i^{2p} - \frac{1}{1+p}\left(R_2^{1+p} - R_1^{1+p}\right)\right) \tag{79}$$

Now, noting that the scalar potential is, in region 1 (radii less than the magnet),

$$\psi = A_1 (r^p - R_i^{2p} r^{-p}) \cos p\theta \qquad r < R_1$$

$$\psi = A_3 (r^p - R_s^{2p} r^{-p}) \cos p\theta \qquad r > R_2$$

and noting that $p(p+1)/(p^2-1) = p/(p-1)$ and $p(p-1)/(p^2-1) = p/(p+1)$, magnetic field is:

$$r < R_1 \tag{80}$$

$$H_r = \frac{M_0}{2\left(R_s^{2p} - R_i^{2p}\right)} \left(\frac{p}{p-1}\left(R_1^{1-p} - R_2^{1-p}\right) R_s^{2p} + \frac{p}{p+1}\left(R_2^{1+p} - R_1^{1+p}\right)\right)\left(r^{p-1} + R_i^{2p} r^{-p-1}\right) \cos p\theta$$

$$r > R_2 \tag{81}$$

$$H_r = \frac{M_0}{2\left(R_s^{2p} - R_i^{2p}\right)} \left(\frac{p}{p-1}\left(R_1^{1-p} - R_2^{1-p}\right) R_i^{2p} + \frac{p}{p+1}\left(R_2^{1+p} - R_1^{1+p}\right)\right)\left(r^{p-1} + R_s^{2p} r^{-p-1}\right) \cos p\theta$$

The case of $p = 1$ appears to be a bit troublesome here, but is easily handled by noting that:

$$\lim_{p \to 1} \frac{p}{p-1}\left(R_1^{1-p} - R_2^{1-p}\right) = \log \frac{R_2}{R_1}$$

Now: there are a number of special cases to consider.

For the iron-free case, $R_i \to 0$ and $R_2 \to \infty$, this becomes, simply, for $r < R_1$:

$$H_r = \frac{M_0}{2} \frac{p}{p-1}\left(R_1^{1-p} - R_2^{1-p}\right) r^{p-1} \cos p\theta \tag{82}$$

Note that for the case of $p = 1$, the limit of this is

$$H_r = \frac{M_0}{2} \log \frac{R_2}{R_1} \cos \theta$$

and for $r > R_2$:

$$H_r = \frac{M_0}{2} \frac{p}{p+1} \left(R_2^{p+1} - R_1^{p+1} \right) r^{-(p+1)} \cos p\theta$$

For the case of a machine with iron boundaries and windings in slots, we are interested in the fields at the boundaries. In such a case, usually, either $R_i = R_1$ or $R_s = R_2$. The fields are:
at the outer boundary: $r = R_s$:

$$H_r = M_0 \frac{R_s^{p-1}}{R_s^{2p} - R_i^{2p}} \left(\frac{p}{p+1} \left(R_2^{p+1} - R_1^{p+1} \right) + \frac{p}{p-1} R_i^{2p} \left(R_1^{1-p} - R_2^{1-p} \right) \right) \cos p\theta$$

or at the inner boundary: $r = R_i$:

$$H_r = M_0 \frac{R_i^{p-1}}{R_s^{2p} - R_i^{2p}} \left(\frac{p}{p+1} \left(R_2^{p+1} - R_1^{p+1} \right) + \frac{p}{p-1} R_s^{2p} \left(R_1^{1-p} - R_2^{1-p} \right) \right) \cos p\theta$$