Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science
6.685 Electric Machines

Problem Set 2 Solutions September 17, 2005

Problem 1: First, note that $\lambda_1 = L_1 i_1 + M i_2 \cos\omega t$, so that

$$v_1 = L_1 \frac{di_1}{dt} + M \cos\omega t \frac{di_2}{dt} - \omega M i_2 \sin\omega t$$

Also, we already know that

$$T^e = -M i_1 i_2 \sin\theta$$

So, for constant $i_2 == I_2$,

$$\begin{aligned} T^e &= -M I_1 I_2 \sin\omega t \\ V_1 &= -\omega M I_2 \sin\omega t \end{aligned}$$

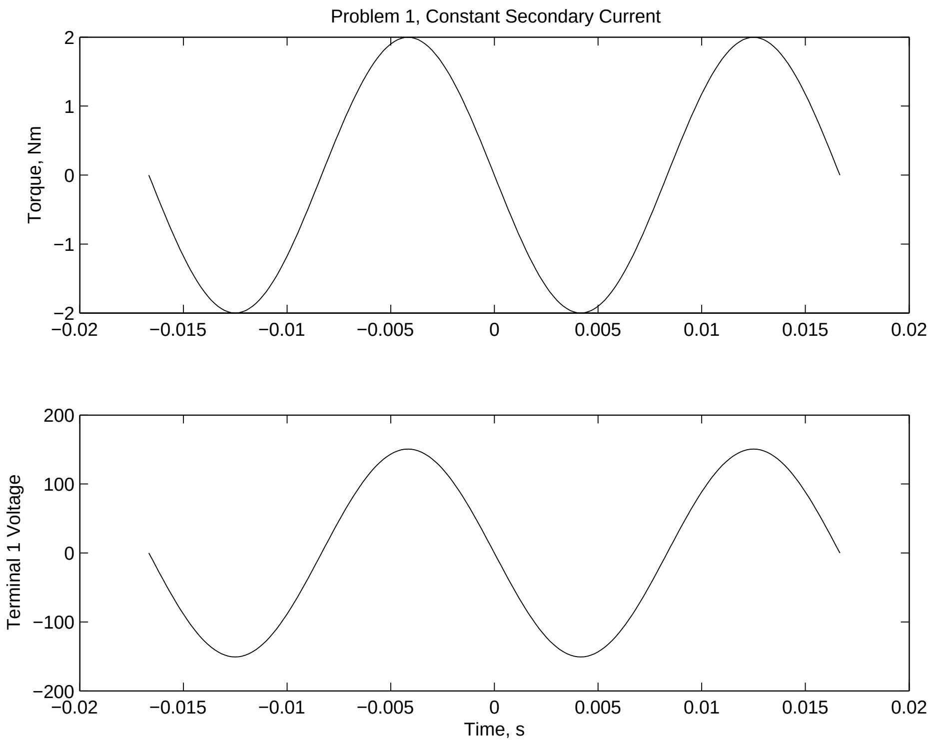

Figure 1: Part 1: Constant Current in Both Coils

Now, if $\lambda_2 = 0$,

$$i_2 = -\frac{M}{L_2} I_1 \cos\omega t$$

and

$$\frac{di_2}{dt} = \omega \frac{MI_1}{L_2} \sin \omega t$$

Torque becomes:

$$T^e = \frac{M^2}{L_2} I_1^2 \sin \omega t \cos \omega t = \frac{M^2}{2L_2} I_1^2 \sin 2\omega t$$

and voltage is

$$v_1 = \omega \frac{M^2 I_1}{L_2} \cos \omega t \sin \omega t + \omega \frac{M^2 I_1}{L_2} \sin \omega t \cos \omega t = \omega \frac{M^2 I_1}{L_2} \sin 2\omega t$$

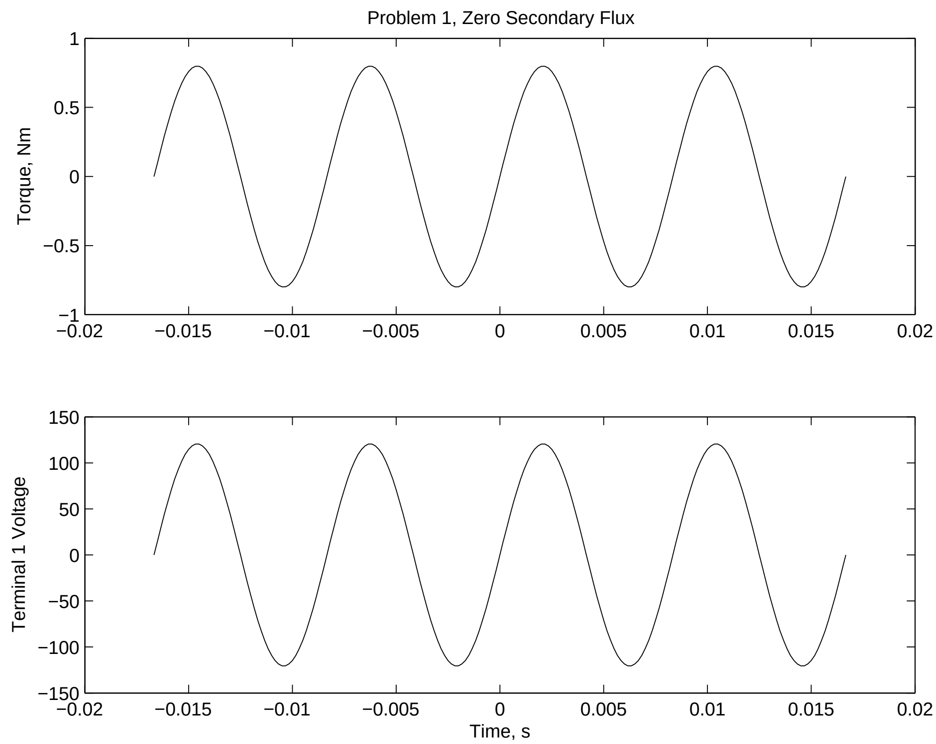

Figure 2: Part 2: Zero Flux in Coil 2

Note that there is no average of voltage or torque and so no real power for these two parts. This makes sense as there is no place for dissipation.

In the third part we do have dissipation. It is convenient, since this is a sinusoidal steady state problem, to use complex variables. Voltage on the secondary side can be written:

$$v_2 = \mathrm{Re} \left\{ \underline{V}_2 e^{j\omega t} \right\}$$

The complex amplitude of v_2 is found by writing KVL:

$$\underline{V}_2 = j\omega M I_1 + j\omega L_2 \underline{I}_2 = -R \underline{I}_2$$

So the complex amplitude of current in the secondary is:

$$\underline{I}_2 = \frac{-j\omega M I_1}{j\omega L_2 + R}$$

The magnitude and angle of the secondary current are easily written:

$$\begin{aligned}
|\underline{I}_2| &= \frac{\omega M I_1}{\sqrt{(\omega L_2)^2 + R^2}} \\
\angle \underline{I}_2 &= -\frac{\pi}{2} - \arctan \frac{\omega L_2}{R} \\
&= -\theta = -\frac{\pi}{2} - \phi
\end{aligned}$$

Torque is then calculated:

$$\begin{aligned}
T^e &= -M i_1 i_2 \sin \omega t \\
&= -M I_1 |\underline{I}_2| \sin \omega t \cos(\omega t - \theta) \\
&= -M I_1 |\underline{I}_2| \left\{ \frac{1}{2} \sin \theta + \frac{1}{2} \sin(2\omega t - \theta) \right\}
\end{aligned}$$

Voltage is, using the chain rule for differentiation:

$$\begin{aligned}
v_1 &= \frac{d\lambda_1}{dt} \\
&= -\omega M i_2 \sin \omega t + M \cos \omega t \frac{di_2}{dt} \\
&= -\omega M |\underline{I}_2| \sin \omega t \cos(\omega t - \theta) - \omega M |\underline{I}_2 \cos \omega t \sin(\omega t - \theta) \\
&= \omega M |\underline{I}_2| \sin(2\omega t - \theta)
\end{aligned}$$

These are plotted in Figure 3

As a check, we might consider the average power input to the machine and the average power dissipated in the resistor, as these must be equal.

Power into the machine through the shaft is easily estimated as speed times torque:

$$P_m = -\omega T^e = \omega M I_1 |\underline{I}_2| \left\{ \frac{1}{2} \sin \theta + \frac{1}{2} \sin(2\omega t - \theta) \right\}$$

Power dissipated in the resistor is

$$P_e = i_2^2 R$$

These are plotted in Figure 4. They appear to have the same average value, but we can prove it easily. See that average electrical power is, substituting for the magnitude of i_2:

$$P_e = \frac{1}{2} |\underline{I}_2|^2 R = \frac{1}{2} \frac{\omega^2 M^2 I_1^2}{(\omega L_2)^2 + R^2} R$$

Average mechanical power in is:

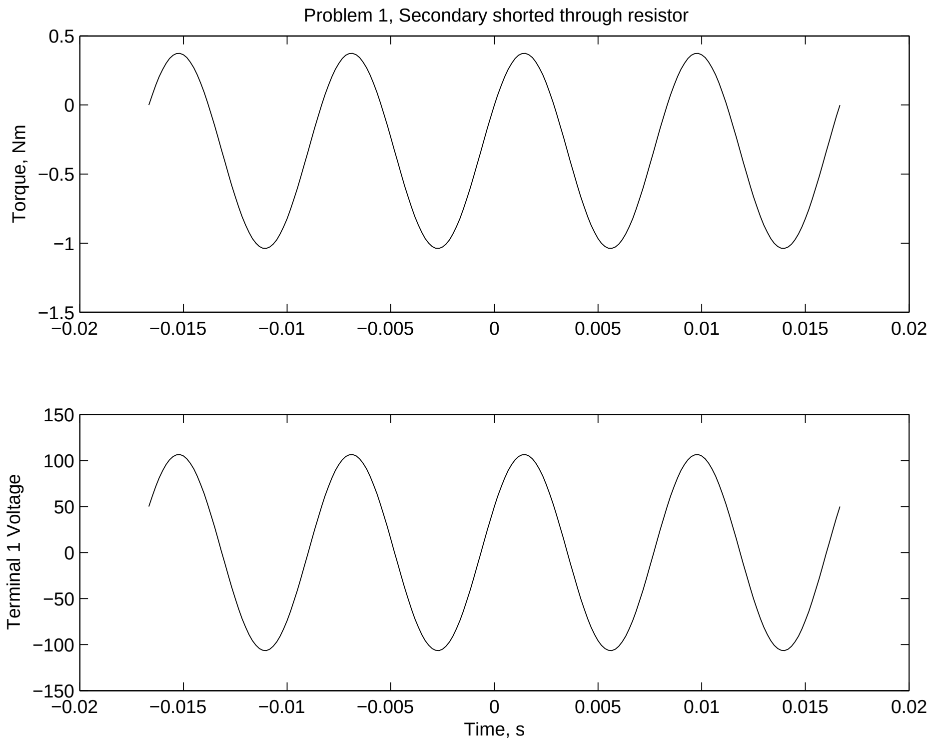

Figure 3: Part 3: Coil 2 shorted through a resistor

$$P_m = -\omega T^e = \frac{\omega^2 M^2}{\sqrt{(\omega L_2)^2 + R^2}} \sin\theta$$

Noting that $\theta = \frac{\pi}{2} + \arctan\frac{\omega L_2}{R}$, we see that

$$\sin\theta = \cos\arctan\frac{\omega L_2}{R} = \frac{R}{\sqrt{(\omega L_2)^2 + R^2}}$$

And making the substitution for $\sin\theta$ into the expression for P_m we have proven that $P_e = P_m$.

Problem 2: Area $A = wD$ is .05 square meter, so force per unit area is

$$P = \frac{F}{A} = \frac{50000N}{.05m^2} = 10^6\text{Pa} = \frac{B^2}{2\mu_0}$$

So that required flux density is

$$B = \sqrt{2\mu_0 P} = 1.5853\text{T}$$

Magentic field is $H = \frac{B}{\mu_0} = 1.2626 \times 10^6\text{A/m}$, And then required ampere-turns are

$$NI = 2gH = 2.5321 \times 10^4$$

Note that this implies a current density in the coil of 252 amperes per square centimeter: a value that would imply the need for good cooling.

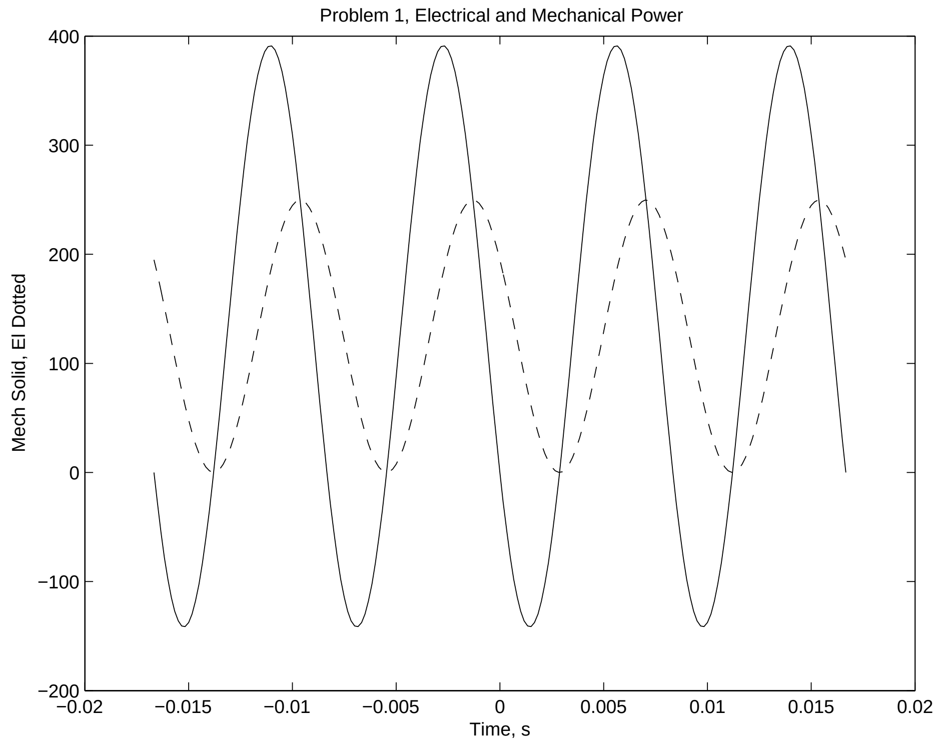

Figure 4: Power in and out for Part 3

Now, if the car is off center, we need to be a step more sophisticated in modeling. Note that the permeances of the two gaps will be:

$$\mathcal{P}_1 = \frac{\mu_0 D}{g}\left(\frac{w}{2} + x\right) \qquad\qquad \mathcal{P}_2 = \frac{\mu_0 D}{g}\left(\frac{w}{2} - x\right)$$

Total permeance is just that of the two gaps in series:

$$\mathcal{P} = \frac{\mathcal{P}_1\mathcal{P}_2}{\mathcal{P}_1 + \mathcal{P}_2} = \frac{\mu_0 D}{gw}\left(\frac{w^2}{4} - x^2\right)$$

Inductance is then simply:

$$L = \frac{\mu_0 N^2 D}{wg}\left(\frac{w^2}{4} - x^2\right)$$

Vertical force is, since this is a linear, singly excited system:

$$F_y = \frac{I^2}{2}\frac{\partial L}{\partial g} = -\frac{\mu_0 (NI)^2 D}{2g^2}\left(\frac{w^2}{4} - x^2\right)$$

Thus required exciting ampere-turns are:

$$(NI)^2 = \frac{wg^2 F_0}{\mu_0 D}\frac{2}{\frac{w^2}{4} - x^2}$$

where F_0 is the required vertical force. This is plotted in Figure 5.

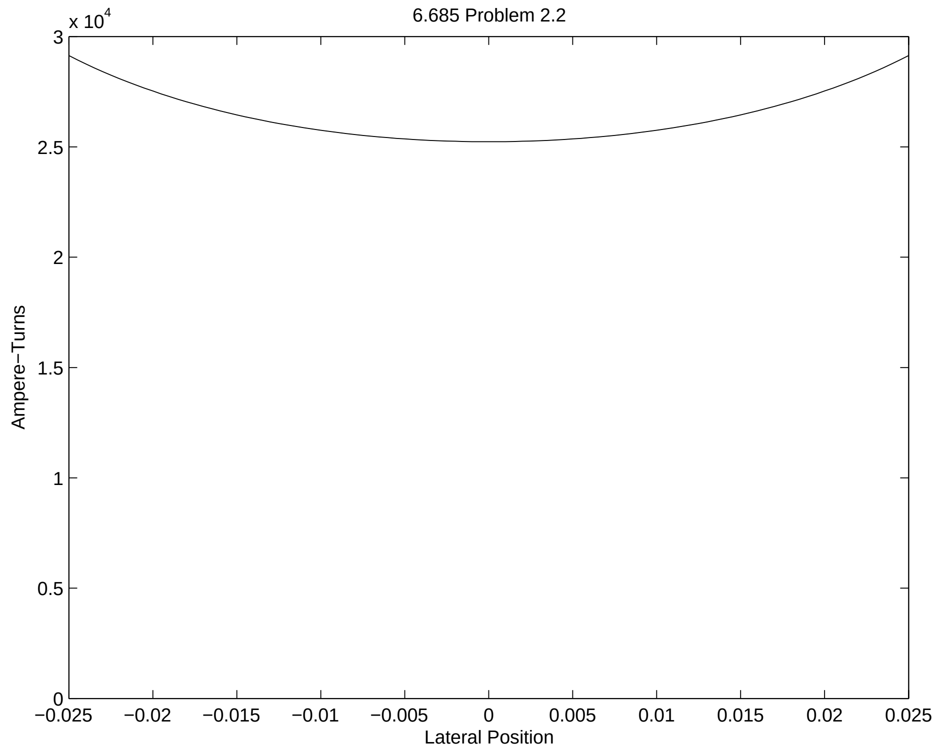

Figure 5: Ampere-Turns required to produce lift force

Lateral force is

$$F_x = \frac{I^2}{2}\frac{\partial L}{\partial x} = -2\frac{\mu_0 \left(NI\right)^2 D}{wg}x$$

When we substitute for the frquired ampere-turns to produce the required vertical force, and doing a little algebra, we find:

$$F_x = -F_0 \frac{4xg}{\frac{w^2}{4} - x^2}$$

This is plotted in Figure 6 for both constant current and for current that must be maintained to produce the correct lift.

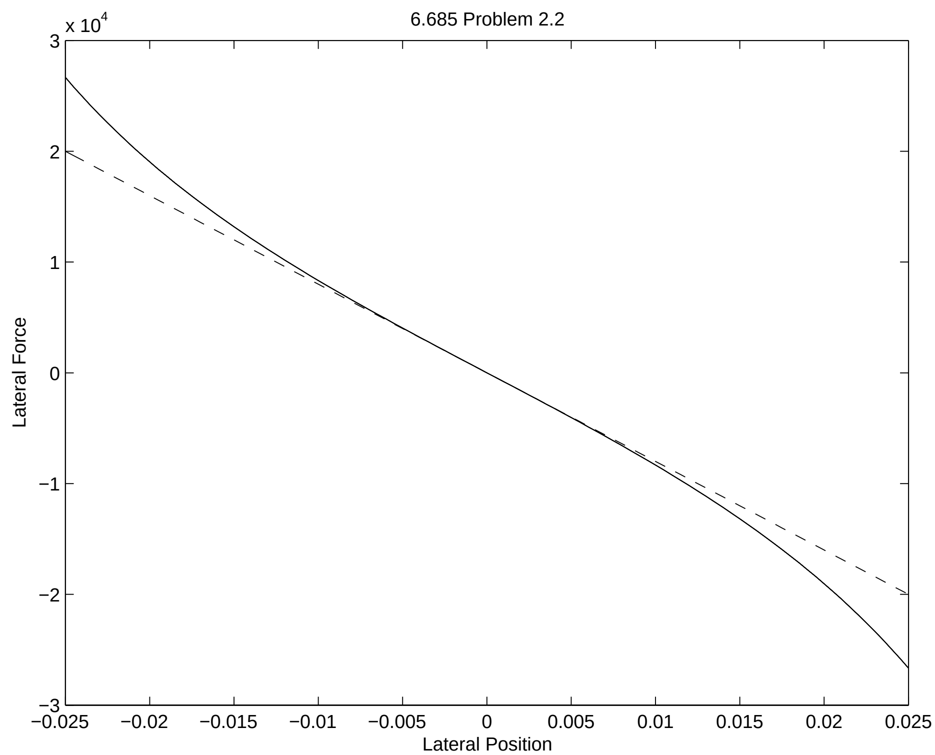

Figure 6: Lateral Force: Constant Current and corrected for current variation

Matlab Script for Problem 1

```
% 6.685 Problem Set 2, Problem 1
% Parameters
L1 = .1;
L2 = .1;
M = .08;
I1 = 5;
R = 20;
om = 2*pi*60;
t =-1/60:1/6000:1/60;

% Part 1: Constant Current
I2 = 5;
T_e1 = -M*I1*I2 .* sin(om .* t);
V11 = -om*M*I2 .* sin(om .* t);

figure(1)
subplot 211
plot(t, T_e1)
title('Problem 1, Constant Secondary Current')
ylabel('Torque, Nm')
subplot 212
plot(t, V11);
ylabel('Terminal 1 Voltage')
xlabel('Time, s')

% Part 2: Rotor shorted

T_e2 = (M^2/(2*L2))*I1^2 .* sin(2*om .*t);
V12 = (om*M^2*I1/L2) .* sin(2*om .*t);

figure(2)
subplot 211
plot(t, T_e2)
title('Problem 1, Zero Secondary Flux')
ylabel('Torque, Nm')
subplot 212
plot(t, V12);
ylabel('Terminal 1 Voltage')
xlabel('Time, s')

% Part 3: Rotor shorted through a resistor

I2m = om*M*I1/sqrt((om*L2)^2+R^2);
phi = atan(om*L2/R);
theta = pi/2 + phi;
```

```
T_e3 = -(M*I1*I2m/2) .* (sin(theta) + sin(2*om .* t - theta));
V13 = -om*M*I2m .* sin(2*om .* t - theta);

figure(3)
subplot 211
plot(t, T_e3)
title('Problem 1, Secondary shorted through resistor')
ylabel('Torque, Nm')
subplot 212
plot(t, V13);
ylabel('Terminal 1 Voltage')
xlabel('Time, s')

% part 4: Power for Part 3

I_2 = I2m .* cos(om .* t - theta);
P2 = R .* I_2 .^2;
Pm = -om .* T_e3;
Pmav = om*M*I1*I2m*sin(theta)/2;
Peav = R * I2m^2/2;
Pma = Pmav .* ones(size(t));
Pea = Peav .* ones(size(t));

figure(4)
subplot 211
plot(t, P2, t, Pea, '--')
title('Problem 1, Electrical and Mechanical Power')
ylabel('Dissipated in Resistor')
subplot 212
plot(t, Pm, t, Pma, '--');
ylabel('Mech Power In')
xlabel('Time, s')

figure(5)
plot(t, P2,'--', t, Pm)
title('Problem 1, Electrical and Mechanical Power')
ylabel('Mech Solid, El Dotted')
xlabel('Time, s')
```

Matlab Script for Problem 2

```
% 6.685 Problem Set 2, Problem 2
% Parameters
D = 0.5;          % length
F = 5e4;          % required force
muzero = pi*4e-7;
g = .01;          % relative motion gap
w = .1;           % pole width, coil width and depth

% First, do the centered case
A = w*D;        % area
B = sqrt(2*muzero*F/A)
N_I = 2*g*B/muzero

x = -.25*w:.005*w:.25*w;

NIS = (2*w*g^2*F/(muzero*D)) ./ (w^2/4 - x .^2);
NI = sqrt(NIS);

figure(1)
plot(x, NI)
axis([-.025 .025 0 3e4])
title('6.685 Problem 2.2')
ylabel('Ampere-Turns')
xlabel('Lateral Position')

% now lateral force
% fixed current
F_xf = -(2*muzero*D*N_I^2/(w*g)) .* x;
% accounting for curent
F_xc = (-2*muzero*D/(w*g)) .* NIS .* x;
F_chk = -(F*4*g) .* x ./((w^2)/4 - x .^2);
figure(2)
plot(x, F_xf,'--', x, F_xc, x, F_chk)
title('6.685 Problem 2.2')
ylabel('Lateral Force')
xlabel('Lateral Position')
```