Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.243j (Fall 2003): DYNAMICS OF NONLINEAR SYSTEMS

by A. Megretski

Take-Home Test 2 Solutions[1]

Problem T2.1

System of ODE equations

$$\dot{x}(t) = Ax(t) + B\phi(Cx(t) + cos(t)), \tag{1.1}$$

where A, B, C are constant matrices such that $CB = 0$, and $\phi : \mathbf{R}^k \mapsto \mathbf{R}^q$ is continuously differentiable, is known to have a locally asymptotically stable non-equilibrium periodic solution $x = x(t)$. What can be said about $\text{trace}(A)$? In other words, find the set Λ of all real numbers λ such that $\lambda = \text{trace}(A)$ for some A, B, C, ϕ such that (1.1) has a locally asymptotically stable non-equilibrium periodic solution $x = x(t)$.

Answer: $\text{trace}(A) < 0$.

Let $x_0(t)$ be the periodic solution. Linearization of (1.1) around $x_0(\cdot)$ yields

$$\dot{\delta}(t) = A\delta(t) + Bh(t)C\delta(t),$$

where $h(t)$ is the Jacobian of ϕ at $x_0(t)$, and

$$x(t) = x_0(t) + \delta(t) + o(|\delta(t)|).$$

Partial information about local stability of $x_0(\cdot)$ is given by the evolution matrix $M(T)$, where $T > 0$ is the period of $x_0(\cdot)$: if the periodic solution is asymptotically stable then all eigenvalues of $M(T)$ have absolute value not larger than one. Here

$$\dot{M}(t) = (A + Bh(t)C)M(t), \quad M(0) = I,$$

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and hence

$$\det M(T) = exp\left(\int_0^T \text{trace}(A + Bh(t)C)dt\right).$$

Since

$$\text{trace}(A + Bh(t)C) = \text{trace}(A + CBh(t)) = \text{trace}(A),$$

$\det(M(T)) > 1$ whenever $\text{trace}(A) > 0$. Hence $\text{trace}(A) \leq 0$ is a necessary condition for local asymptotic stability of $x_0(\cdot)$.

Since system (1.1) with $k = q = 1$, $\phi(y) \equiv y$,

$$A = \begin{bmatrix} -a & 0 \\ 0 & -a \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

has periodic stable steady state solution

$$x_0(t) = \begin{bmatrix} (1 + a^2)^{-1}\cos(t) + a(1 + a^2)^{-1}\sin(t)) \\ 0 \end{bmatrix}$$

for all $a > 0$, the trace of A can take every negative value. Thus, to complete the solution, one has to figure out whether trace of A can take the zero value.

It appears that the volume contraction techniques are better suited for solving the question completely. Indeed, consider the autonomous ODE

$$\left\{\begin{array}{rcl} \dot{z}_1(t) & = & z_2(t), \\ \dot{z}_2(t) & = & -z_1(t), \\ \dot{z}_3(t) & = & Az_3(t) + B\phi\left(Cz_3(t) + \frac{z_1(t)}{\sqrt{z_1(t)^2 + z_2(t)^2}}\right), \end{array}\right. \tag{1.2}$$

defined for $z_1^2 + z_2^2 \neq 0$. If (1.1) has an asymptotically stable periodic solution $x_0 = x_0(t)$ then, for $\epsilon > 0$ small enough, solutions of (1.2) with

$$\left\| \begin{bmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \end{bmatrix} - \bar{z} \right\| \leq \epsilon, \quad \bar{z} \begin{bmatrix} 1 \\ 0 \\ x_0(0) \end{bmatrix}$$

small enough satisfy

$$\lim_{t \to \infty} z_3(t) - x_0(t + \tau) = 0,$$

where $\tau \approx 0$ is defined by $z_2(-\tau) = 0$. In particular, the Euclidean volume of the image of the the ball of radius ϵ centered at $\bar{z}$ under the differential flow defined by (1.2) converges to zero as $t \to \infty$. Since the volume is non-increasing when $\text{trace}(A) \geq 0$, we conclude that $\text{trace}(A) < 0$.

Problem T2.2

FUNCTION $g_1: \mathbf{R}^3 \mapsto \mathbf{R}^3$ IS DEFINED BY

$$g_1\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ x_1 \\ 0 \end{bmatrix}.$$

(a) FIND A CONTINUOUSLY DIFFERENTIABLE FUNCTION $g_2: \mathbf{R}^3 \mapsto \mathbf{R}^3$ SUCH THAT THE DRIFTLESS SYSTEM

$$\dot{x}(t) = g_1(x(t))u_1(t) + g_2(x(t))u_2(t) \tag{1.3}$$

IS COMPLETELY CONTROLLABLE ON $\mathbf{R}^3$.

For

$$g_2(x) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \text{const},$$

we have

$$g_3 = [g_1, g_2] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Since $g_1(x), g_2, g_3$ form a basis in $\mathbf{R}^3$ for all x, the resulting system (1.3) is completely controllable on $\mathbf{R}^3$.

(b) FIND CONTINUOUSLY DIFFERENTIABLE FUNCTIONS $g_2: \mathbf{R}^3 \mapsto \mathbf{R}^3$ AND $h: \mathbf{R}^3 \mapsto \mathbf{R}$ SUCH THAT $\nabla h(\bar{x}) \neq 0$ FOR ALL $\bar{x} \in \mathbf{R}^3$ AND $h(x(t))$ IS CONSTANT ON ALL SOLUTIONS OF (1.3). (NOTE: FUNCTION g_2 IN (B) DOES NOT HAVE TO BE (AND CANNOT BE) THE SAME AS g_2 IN (A).)

For example,

$$g_2(x) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \text{const}, \quad h\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = x_3.$$

(c) FIND A CONTINUOUSLY DIFFERENTIABLE FUNCTION $g_2: \mathbf{R}^3 \mapsto \mathbf{R}^3$ SUCH THAT THE DRIFTLESS SYSTEM (1.3) IS NOT COMPLETELY CONTROLLABLE ON $\mathbf{R}^3$, BUT, ON THE OTHER HAND, THERE EXISTS NO CONTINUOUSLY DIFFERENTIABLE FUNCTION $h: \mathbf{R}^3 \mapsto \mathbf{R}$ SUCH THAT $\nabla h(\bar{x}) \neq 0$ FOR ALL $\bar{x} \in \mathbf{R}^3$ AND $h(x(t))$ IS CONSTANT ON ALL SOLUTIONS OF (1.3).

For

$$g_2(x) = \begin{bmatrix} 0 \\ x_1 \\ x_3 \end{bmatrix},$$

we have

$$g_3 = [g_2, g_1] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and hence $g_1(x), g_2, g_3$ form a basis in $\mathbf{R}^3$ whenever $x_3 \neq 0$. This contradicts the condition that $\nabla h(x)$ must be non-zero ad orthogonal to $g_1(x), g_2$ (and hence to g_3) for all x.

Problem T2.3

An ODE control system model is given by equations

$$\begin{cases} \dot{x}_1(t) &= x_2(t)^2 + u(t), \\ \dot{x}_2(t) &= x_3(t)^2 + u(t), \\ \dot{x}_3(t) &= p(x_1(t)) + u(t). \end{cases} \tag{1.4}$$

(a) Find all polynomials $p: \mathbf{R} \mapsto \mathbf{R}$ such that system (1.4) is full state feedback linearizable in a neigborhood of $\bar{x} = 0$.

System (1.4) has the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \tag{1.5}$$

where

$$f\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_2^2 \\ x_3^2 \\ p(x_1) \end{bmatrix}, \quad g\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Define

$$g_1 = g, \quad g_2 = [f, g_1], \quad g_3 = [f, g_2], \quad g_{21} = [g_2, g_1],$$

i.e.

$$g_2\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_2 \\ 2x_3 \\ \dot{p}(x_1) \end{bmatrix}, \; g_3\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 4x_2x_3 - 2x_3^2 \\ 2x_3\dot{p}(x_1) - 2p(x_1) \\ 2x_2\dot{p}(x_1) - \ddot{p}(x_1)x_2^2 \end{bmatrix}, \; g_{21}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ \ddot{p}(x_1) \end{bmatrix}.$$

For local full state feedback linearizability at $x = 0$ it is necessary and sufficient for vectors $g_1(0), g_2(0), g_3(0)$ to be linearly independent (which is equivalent to $p(0)\dot{p}(0) = 0$) and for $g_{21}(x)$ to be a linear combination of $g_1(x)$ and $g_2(x)$ for *all* x in a neigborhood of $x = 0$ (which is equivalent to $\ddot{p}(x_1) \equiv 2$). Hence

$$p(x_1) = x_1^2 + p_1x_1 + p_0, \quad p_0p_1 \neq 0$$

is necessary and sufficient for local full state feedback linearizability at $x = 0$.

(b) FOR EACH POLYNOMIAL p FOUND IN (A), DESIGN A FEEDBACK LAW

$$u(t) = K(x_1(t), x_2(t), x_3(t)) = K_p(x_1(t), x_2(t), x_3(t))$$

WHICH MAKES THE ORIGIN A LOCALLY ASYMPTOTICALLY STABLE EQUILIBRIUM OF (1.4).

Since $p(0) \neq 0$, $x = 0$ cannot be made into a locally asymptotically stable equilibrium of (1.4). However, the origin $z = 0$ (i.e. with respect to the new coordinates $z = \psi(x)$) of the feedback linearized system can be made locally asymptotically stable, as long as $0 \in \psi(\Omega)$ where Ω is the domain of ψ. Actually, this does not require any knowledge of the coordinate transform ψ, and can be done under an assumption substantially weaker than full state feedback linearizability!

Let

$$\dot{z}(t) = Az(t) + Bv(t) \tag{1.6}$$

be the feedback linearized equations (1.5), where

$$z(t) = \psi(x(t)), \quad x(t) \in \Omega, \quad v(t) = \alpha(x(t))(u - \beta(x(t))).$$

In other words, let

$$f(x) = [\dot{\psi}(x)]^{-1}[A\psi(x) - B\alpha(x)\beta(x)], \quad g(x) = [\dot{\psi}(x)]^{-1}B\alpha(x).$$

If $\bar{x} \in \Omega$ satisfies $\psi(\bar{x}) = 0$ then $\bar{x}$ is a conditional equilibrium of (1.5), in the sense that

$$f(\bar{x}) + g(\bar{x})\bar{u} = 0$$

for $\bar{u} = \beta(\bar{x})$. Moreover, since the pair (A, B) is assumed to be controllable, the conditional equilibrium has a controllable linearization, in the sense that the pair $(\dot{f}(\bar{x}) + \dot{g}(\bar{x})\bar{u}, g(\bar{x}))$ is controllable as well, because

$$\dot{f}(\bar{x}) + \dot{g}(\bar{x})\bar{u} = S^{-1}(AS - BF), \; g(\bar{x}) = S^{-1}B\alpha(\bar{x})$$

for

$$S = \dot{\psi}(\bar{x}), \; F = \alpha(\bar{x})\dot{\beta}(\bar{x}).$$

It is easy to see that *every* conditional equilibrium $\bar{x}$ of (1.5) with a controllable linearization can be made into a locally exponentially stable equilibrium by introducing feedback control

$$u(t) = \bar{u} + K(x(t) - \bar{x}),$$

where K is a constant gain matrix such that

$$\dot{f}(\bar{x}) + \dot{g}(\bar{x})\bar{u} + g(\bar{x})K$$

is a Hurwitz matrix. Indeed, by assumption $\bar{x}$ is an equilibriun of

$$\dot{x}(t) = f_K(x) = f(x(t)) + g(x(t))(\bar{u} + K(x(t) - \bar{x})),$$

and

$$\dot{f}_K(\bar{x}) = \dot{f}(\bar{x}) + \dot{g}(\bar{x})\bar{u} + g(\bar{x})K.$$

In the case of system (1.4) let

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix}$$

be a conditional equilibrium, i.e.

$$\bar{x}_1^2 = \bar{x}_2^2 = p(\bar{x}_1) = -\bar{u}.$$

Then

$$\dot{f}(\bar{x}) + \dot{g}(\bar{x})\bar{u} = \begin{bmatrix} 0 & \bar{x}_2 & 0 \\ 0 & 0 & 2\bar{x}_2 \\ \dot{p}(\bar{x}_1) & 0 & 0 \end{bmatrix}, \quad g(\bar{x}) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Hence a locally stabilizing controller is given by

$$u(t) = -\bar{x}_1^2 + k_1(x_1(t) - \bar{x}_1) + k_2(x_2(t) - \bar{x}_2) + k_3(x_3(t) - \bar{x}_3),$$

where the coefficients k_1, k_2, k_3 are chosen in such a way that

$$\begin{bmatrix} 0 & \bar{x}_2 & 0 \\ 0 & 0 & 2\bar{x}_2 \\ \dot{p}(\bar{x}_1) & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

is a Hurwitz matrix.

(c) FIND A C^∞ FUNCTION $p: \mathbf{R} \mapsto \mathbf{R}$ FOR WHICH SYSTEM (1.4) IS GLOBALLY FULL STATE FEEDBACK LINEARIZABLE, OR PROVE THAT SUCH $p(\cdot)$ DOES NOT EXIST.

Such $p(\cdot)$ does not exist. Indeed, otherwise vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2x_2 \\ 2x_3 \\ \dot{p}(x_1) \end{bmatrix}$$

are linearly independent for all real $\bar{x}_1, \bar{x}_2, \bar{x}_3$, which is impossible for

$$\bar{x}_2 = \bar{x}_3 = 0.5\dot{p}(\bar{x}_1).$$