**Massachusetts Institute of Technology**

**Department of Electrical Engineering and Computer Science**

**6.243j (Fall 2003): DYNAMICS OF NONLINEAR SYSTEMS**

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# Problem Set 1[1]

## Problem 1.1

Behavior set $\mathcal{B}$ of an autonomous system with a scalar binary DT output consists of all DT signals $w = w(t) \in \{0, 1\}$ which change value at most once for $0 \le t < \infty$.

(a) Give an example of two signals $w_1, w_2 \in \mathcal{B}$ which commute at $t = 3$, but do not define same state of $\mathcal{B}$ at $t = 3$.

(b) Give an example of two *different* signals $w_1, w_2 \in \mathcal{B}$ which define same state of $\mathcal{B}$ at $t = 4$.

(c) Find a time-invariant discrete-time finite state-space "difference inclusion" model for $\mathcal{B}$, i.e. find a *finite* set $X$ and functions $g : X \mapsto \{0, 1\}$, $f : X \mapsto S(X)$, where $S(X)$ denotes the set of all non-empty subsets of $X$, such that a sequence $w(0), w(1), w(2), \ldots$ can be obtained by sampling a signal $w \in \mathcal{B}$ if and only if there exists a sequence $x(0), x(1), x(2), \ldots$ of elements from $X$ such that

$$x(t+1) \in f(x(t)) \quad \text{and} \quad w(t) = g(x(t)) \quad \text{for } t = 0, 1, 2, \ldots.$$

(Figuring out which pairs of signals define same state of $\mathcal{B}$ at a given time is one possible way to arrive at a solution.)

## Problem 1.2

Consider differential equation

$$\ddot{y}(t) + \text{sgn}(\dot{y}(t) + y(t)) = 0.$$

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(a) Write down an equivalent ODE $\dot{x}(t) = a(x(t))$ for the state vector $x(t) = [y(t); \dot{y}(t)]$.

(b) Find all vectors $x_0 \in \mathbf{R}^2$ for which the ODE from (a) does not have a solution $x : [t_0, t_1] \mapsto \mathbf{R}^2$ (with $t_1 > t_0$) satisfying initial condition $x(t_0) = x_0$.

(c) Define a semicontinuous convex set-valued function $\eta : \mathbf{R}^2 \mapsto 2^{\mathbf{R}^2}$ such that $a(\bar{x}) \in \eta(\bar{x})$ for all $x$. Make sure the sets $\eta(\bar{x})$ are the smallest possible subject to these constraints.

(d) Find explicitly all solutions of the differential inclusion $\dot{x}(t) \in \eta(x(t))$ satisfying initial conditions $x(0) = x_0$, where $x_0$ are the vectors found in (b). Such solutions are calles *sliding modes*.

(e) Repeat (c) for $a : \mathbf{R}^2 \mapsto \mathbf{R}^2$ defined by

$$a([x_1; x_2]) = [\mathrm{sgn}(x_1); \mathrm{sgn}(x_2)].$$

**Problem 1.3**

For the statements below, state whether they are true or false. For true statements, give a *brief* proof (can refer to lecture notes or books). For false statements, give a counterexample.

(a) All maximal solutions of ODE $\dot{x}(t) = \exp(-x(t)^2)$ are defined on the whole time axis $\{t\} = \mathbf{R}$.

(b) All solutions $x : \mathbf{R} \mapsto \mathbf{R}$ of the ODE

$$\dot{x}(t) = \begin{cases} x(t)/t, & t \neq 0, \\ 0, & t = 0 \end{cases}$$

are such that $x(t) = -x(-t)$ for all $t \in \mathbf{R}$.

(c) If constant signal $w(t) \equiv 1$ belongs to a system behavior set $\mathcal{B}$, but constant signal $w(t) \equiv -1$ does not then the system is not linear.