6.241: Dynamic Systems—Spring 2011

HOMEWORK 10 SOLUTIONS

Exercise 23.1 a) We are given the single input LTI system:

$$\dot{x} = Ax + bu \ , \ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ , \ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The solution is expressed by:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}bu(\tau)d\tau$$

Calculate exponent of matrix A by summing up the series and taking into account that $A^n = 0$, $\forall n > 1$.

$$e^{At} = I + At = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

thus

$$e^{At} b = \begin{bmatrix} t \\ 1 \end{bmatrix}$$

b) the reachability matrix is:

$$\begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The reachability matrix has rank 2, therefore the system is reachable. Now, we compute the reachability Grammian over an interval of length 1:

$$G = \int_0^1 e^{A(T-\tau)}bb'e^{A(T-\tau)'}d\tau = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

The system is reachable thus the Grammian is invertible, so given any final state x_f we can always find α such that $x_f = G\alpha$. In particular

$$\alpha = \frac{1}{\sqrt{2}} \begin{bmatrix} 18 \\ -10 \end{bmatrix}$$

c) According to 23.5 define $F^T(t) = e^{A(1-t)} b$. Then $u(t) = F(t)\alpha$ is a control input that produces a trajectory that satisfies the terminal constraint x_f. The control effort is given as:

$$\int_0^T u^2 d\tau = \alpha' \ G \ \alpha$$

Infact this input corresponds to the minimum energy input required to reach x_f in 1 second. This can be verified by solving the corresponding underconstrained least squares problem by means of the tools we learned at chapter 3.

d) First of all note that

$$\alpha' \, G \, \alpha = x'_f \, G^{-1} \, x_f$$

The Grammian as well as its inverse are symmetric matrices. If we want to maximize the energy, $\max\{x'_f G^{-1} x_f | \, \|x_f\| = 1\}$, we have to choose x_f alligned with the singular vector corresponding to $\sigma_{min}(G)$.

Exercise 23.4 Given :

$$\dot{x}(t) = Ax + (b + \delta)u,$$

where $\delta \in \mathbb{R}^n$, and (A, b) is reachable.
a) Using the Theorem 22.2, in order to make the system unreachable, we have $w^T B =$ for some left eigenvectors w^T of A. So, let λ_i is an eigenvalue of A and w_i be the corresponding left eigenvectors. Then, using the theorem, we want to find δ which makes this eigenmode unreachable $\leftrightarrow w_i^T(b + \delta) = 0$. So, now we have

$$w_i^T \delta = -w_i^T b.$$

Then with this constraint, we would like to minimize $\|\delta\|_2$. Thus this can be cast into an optimization problem as follows:

$$
\begin{aligned}
\text{Find} \quad & min\|\delta\|_2 \\
s.t. \quad & w_i^T \delta = -w_i^T b.
\end{aligned}
$$

This is exactly in the form of the least square problem. Since both δ and b are real, even when $w_i \in \mathbb{C}^n$, let $\tilde{w}_i = \begin{bmatrix} w_i^R & w_i^I \end{bmatrix}$, where w_i^R and w_i^I are real and imaginary parts of w_i respectively. Then the formulation still remains as a least square problem as follows:

$$
\begin{aligned}
\text{Find} \quad & \|\delta\|_2 \\
s.t. \quad & \tilde{w}_i^T \delta = \tilde{w}_i^T b.
\end{aligned}
$$

Then the solution to this problem is

$$
\begin{aligned}
\hat{\delta} \quad &= \quad -\tilde{w}_i(\tilde{w}_i^T \tilde{w}_i)^{-1}\tilde{w}_i^T b \\
\therefore \min \|\delta\|_2 \quad &= \quad \sqrt{\hat{\delta}^T \hat{\delta}}
\end{aligned}
$$

The last expression has to be minimized over all possible left eigenvectors of A. Note that the expression does not depend on the norm of the eigenvectors, thus we can minimize over eigenvectors with unity norm. If all Jordan blocks of matrix A have different eigenvalues, this is a minimization over a finite set. In the other case we can represent eigenvectors corresponding to Jordan blocks with the same eigenvalues as a linear combination of eigenvectors corresponding to particular Jordan blocks, and then minimize over the coefficients in the linear combination.

b) NO. The explanation is as follows. With the control suggested, the closed loop dynamics is now

$$
\begin{aligned}
\dot{x} &= Ax + (b + \delta)u \\
u &= f^T x + v \\
\rightarrow \dot{x} &= (A + (b + \delta)f^T)x + (b + \delta)v.
\end{aligned}
$$

Suppose that w_i was the minimizing eigenvector of unity norm in part a). Then it is also an eigenvector of matrix $A + (b + \delta)f^T$ since w_i is orthogonal to $b + \delta$. Therefore feedback does not improve reachability.

Exercise 24.5 a) The given system in general for all $t \geq 0$ with $u(k) = 0 \; \forall k \geq 0$ has the following expression for the output:

$$
\begin{aligned}
y(t) &= \Sigma_{k=-1}^{-\infty} CA^{t-k-1}Bu(k) \\
&= CA^t \Sigma_{k=-1}^{-\infty} A^{-k-1}Bu(k)
\end{aligned}
$$

since matrix A is stable. Note that because of stability of matrix A all of its eigenvalues are strictly within unit circle, and from Jordan decomposition we can see that

$$
\lim_{k \to \infty} \|A^k\|_2 = 0
$$

therefore $x(-\infty)$ does not influence $x(0)$. Thus the above equation can be used in order to find $x(0)$ as follows:

$$
x(0) = \sum_{k=-1}^{-\infty} A^{-k-1}Bu(k).
$$

b) Since the system is reachable, any $\xi \in \mathbf{R}^n$ can be achieved by some choice of an input of the above form. Also, since the system is reachable, the reachability matrix $\mathcal{R}$ has full row rank. As a consequence $(\mathcal{R}\mathcal{R}^T)^{-1}$ exists. Thus, in order to minimize the input energy, we have to solve the following familiar least square problem:

$$
\begin{aligned}
\text{Find} \quad &\min \|u\|_2 \\
s.t. \quad &\xi = \sum_{k=-1}^{-\infty} A^{-k-1}Bu(k).
\end{aligned}
$$

Then the solution can be written in terms of the reachability matrix as follows:

$$
u_{min} = \mathcal{R}^T (\mathcal{R}\mathcal{R}^T)^{-1}\xi,
$$

so that its square can be expressed as

$$
\begin{aligned}
\|u\|_{min}^2 &= u_{min}^T u_{min} \\
&= \xi^T (((\mathcal{R}\mathcal{R}^T)^{-1})^T \mathcal{R}\mathcal{R}^T (\mathcal{R}\mathcal{R}^T)^{-1}\xi \\
&= \xi^T (\mathcal{R}\mathcal{R}^T)^{-1}\xi,
\end{aligned}
$$

where the last equality comes from the fact that inverse of a symmetric positive definite matrix is still symmetric positive definite. Also, the Controlability Gramian of DT systems $\mathcal{P}$ is

$$\mathcal{P} = \sum_{k=0}^{\infty} A^k BB^T (A^T)^k = \mathcal{R}\mathcal{R}^T,$$

and is symmetric positive definite. Thus the square of the minimum energy, denoted as $\alpha_1(\xi)$, can be expressed as

$$\alpha_1(\xi) = \xi^T \mathcal{P}^{-1} \xi = \|M\xi\|_2^2$$

where M is a Hermitian square root matrix of $\mathcal{P}^{-1}$ which is still symmetric positive definite.

c) Suppose some input u_{min} results in $x(0) = \xi$, then the output for $t \geq 0$ can be expressed as

$$y(t) = Cx(t) = CA^t \xi.$$

Thus the square of the energy of the output for $t \geq 0$ can be written as

$$
\begin{aligned}
\|y\|_2^2 &= (y^T y) \\
&= \left(\begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix} \xi \right)^T \begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix} \xi \\
&= \xi^T \left(\sum_{t=0}^{\infty} (A^T)^k C^T C A^k \right) \xi \\
&= \xi^T \mathcal{O}^T \mathcal{O} \xi
\end{aligned}
$$

Since the Observability Grammian of DT systems $\mathcal{Q}$ is

$$\mathcal{Q} = \sum_{k=0}^{\infty} (A^T)^k C^T C A^k = \mathcal{O}^T \mathcal{O},$$

the square of the energy of the output for $t \geq 0$, which we now denote $\alpha_2(\xi)$, can be expressed as a function of ξ as follows:

$$\alpha_2(\xi) \equiv \xi^T \mathcal{Q} \xi.$$

Also, because of the symmetric positive definiteness of $\mathcal{Q}$, $\alpha_2(\xi)$ can be written as

$$\alpha_2(\xi) = \|N\xi\|_2^2,$$

where N is a Hermitian square root matrix of $\mathcal{Q}$.

d) It can be argued as follows:

4

$$
\begin{aligned}
\alpha &= \max_u\left\{\sum_{t=0}^{\infty} y(t)^2 \;\Big|\; \sum_{t=-\infty}^{-1} u(t)^2 \leq 1 \;,\; u(k) = 0 \forall k \geq 0\right\} \\
&= \max_\xi\left\{\alpha_2(\xi) \;\Big|\; \exists\, u \text{ s.t.} \xi = x(0) \text{ and } \sum_{t=-\infty}^{-1} u(t)^2 \leq 1 \;,\; u(k) = 0 \;,\; \forall k \geq 0\right\} \\
&= \max_\xi\left\{\alpha_2(\xi) \;\big|\; \|u_{min}\|_2^2 \leq 1\right\} \\
&= \max_\xi\left\{\alpha_2(\xi) \;\big|\; \alpha_1(\xi) \leq 1\right\}.
\end{aligned}
$$

e) Now, using the fact shown in d) and noting the fact that $\mathcal{P}^{-1} = M^T M$ where M is Hermitian square root matrix which is invertible, we can compute α:

$$
\begin{aligned}
\alpha &= \max_\xi\left\{\alpha_2(\xi) \;\big|\; \alpha_1(\xi) \leq 1\right\} \\
&= \max_\xi\left\{\|N\xi\|_2^2 \;\big|\; \|M\xi\|_2^2 \leq 1\right\} \ \text{ set } \xi = M^{-1}l \\
&= \max_l\left\{(M^{-1}l)^T \mathcal{O}^T \mathcal{O} M^{-1}l \;\big|\; \|l\|_2^2 \leq 1\right\} \\
&= \sigma_{max}(\mathcal{O}M^{-1}) \\
&= \lambda_{max}((M^{-1})^T \mathcal{O}^T \mathcal{O} M^{-1}) \\
&= \lambda_{max}((M^{-1})^T \mathcal{Q} M^{-1}) \\
&= \lambda_{max}(\mathcal{Q} M^{-1}(M^{-1})^T) \\
\therefore \alpha &= \lambda_{max}(\mathcal{Q}\,\mathcal{P})
\end{aligned}
$$

Exercise 25.2 a) Given:

$$
H_1(s) = \frac{s+f}{(s+4)^3} = \frac{s+f}{s^3 + 12s^2 + 48s + 64} \;,\; H_2(s) = \frac{1}{s-2}.
$$

Thus the state-space realizations in controller canonical form for $H_1(s)$ and $H_2(s)$ are :

$$
A_1 = \begin{pmatrix} -12 & -48 & -64 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \;,\; B_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \;,\; C_1 = \begin{pmatrix} 0 & 1 & f \end{pmatrix} \;,\; D_1 = 0,
$$

and

$$
A_2 = 2 \;,\; B_2 = 1 \;,\; C_2 = 1 \;,\; D_2 = 0.
$$

Since f is not included in the controllability matrix for $H_1(s)$ with this realization, the controllability, which is equivalent to reachability for CT cases, the controllability is independent of the value of f.Thus, check the rank of the controllability matrix:

$$rank(\mathcal{C}) \;=\; rank \begin{pmatrix} 1 & -12 & 96 \\ 0 & 1 & -12 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \; 3.$$

Thus, the system with this realization is controllable. On the other hand, the observability matrix $\mathcal{O}$ for $H_1(s)$ contains f in it as follows:

$$\mathcal{O} = \begin{pmatrix} 0 & 1 & f \\ 1 & f & 0 \\ -12 + f & -48 & -60 \end{pmatrix}.$$

Thus, when $f = 4$, $\mathcal{O}$ decreases its rank from 3 to 2.

Now, let's consider the state-space realization in observer canonical form for $H_1(s)$. It can be expressed as follows:

$$A_1 = \begin{pmatrix} 0 & 0 & -64 \\ 1 & 0 & -48 \\ 0 & 1 & -12 \end{pmatrix} \;,\; B_1 = \begin{pmatrix} f \\ 1 \\ 0 \end{pmatrix} \;,\; C_1 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \;,\; D_1 = 0.$$

Since C_1 does not contain f, the observability in independent of the value f. Thus check the rank of the observability matrix:

$$rank(\mathcal{O}) \;=\; rank \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -12 \\ 1 & -12 & 96 \end{pmatrix}$$

$$= \; 3.$$

Thus thus the system with this realization is observable.
On the other hand, the controllability matrix contains f in it as follows:

$$\mathcal{C} = \begin{pmatrix} f & 0 & -64 \\ 1 & f & -48 \\ 0 & 1 & f-12 \end{pmatrix}.$$

Thus, again when $f = 4$, $\mathcal{C}$ decreases its rank from 3 to 2.

b) Let $H(s)$ be the cascaded system, $H_2(s)H_1(s)$. Then, the augmented system $H(s)$ has the following state-space representation:

6

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u \\ y = \begin{pmatrix} 0 & C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{cases}$$

$$\begin{cases} \underline{\dot{x}} = \begin{pmatrix} -12 & -48 & -64 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & f & 2 \end{pmatrix} \underline{x} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} u \\ y = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \underline{x} \end{cases}$$

$$\rightarrow \begin{cases} \dot{x} = Ax + Bu \\ y = Cx. \end{cases}$$

Here, we use A_1, B_1, and C_1 from the controller canonical form obtained in a). Since matrix A has zero block in its upper triangle, the eigenvalues of the cascaded system are ones of A_1 and A_2, i.e., $-4, -4, -4$, and 2. Thus the cascaded system is not asymptotically stable. Since C_1 is not included in the eigenvalue computation for A, the stability does not depend on the value of f.
The controllability matrix $\mathcal{C}$ for $H(s)$ is

$$\begin{aligned} \mathcal{C} &= \begin{pmatrix} B & AB & A^2B & A^3B \end{pmatrix} \\ &= \begin{pmatrix} 1 & -12 & 12^2 - 48 & -12^3 + 48*12*2 - 64 \\ 0 & 1 & -12 & 12^2 - 48 \\ 0 & 0 & 1 & -12 \\ 0 & 0 & 1 & -12 + f + 2 \end{pmatrix}, \end{aligned}$$

which decreases its rank from 4 to 3 when $f = -2$. On the other hand, the observability matrix $\mathcal{O}$ for $H(s)$ is

$$\begin{aligned} \mathcal{O} &= \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & f & 2 \\ 1 & f+2 & 2f & 4 \\ -12 + f + 2 & -48 + 2f + 4 & -64 + 4f & 8 \end{pmatrix}, \end{aligned}$$

thus the choice of $f = 4$, $\mathcal{O}$ drops its rank from full rank to 3. Thus the cascaded system is unoberservable at $f = 4$.
It can be seen immediately that $f = 2$ case corresponds to unstable pole-zero cancellation. Thus, for $f = 2$, the cascaded system is BIBO stable, but is not asymptotically stable due to the unstable pole-zero cancellation.

6.241J / 16.338J Dynamic Systems and Control

Spring 2011