Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.341: Discrete-Time Signal Processing

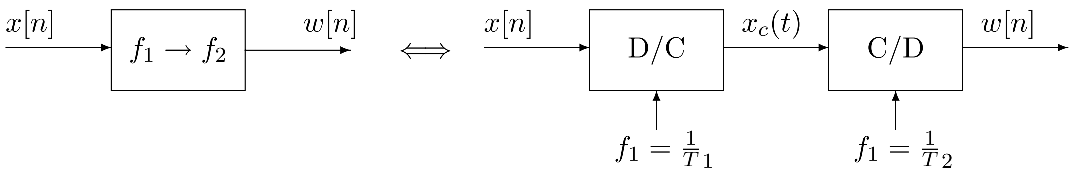
OpenCourseWare 2006

Lecture 5

Sampling Rate Conversion

Reading: Section 4.6 in Oppenheim, Schafer & Buck (OSB).

It is often necessary to change the sampling rate of a discrete-time signal to obtain a new discrete-time representation of the underlying continuous-time signal. The desired system is shown below:



Sample rate converter

Sampling Rate Compression by an Integer Factor

To reduce the sampling rate of a sequence by an integer factor, the sequence can be further compressed or decimated as depicted in OSB Figure 4.20. This discrete-time sampler can be interpreted as the cascade of a D/C converter and a C/D converter in which:

$$x[n] = x_c(nT) \ ,$$

$$x_d[n] = x[nM] = x_c(nMT).$$

The discrete-time Fourier transform of $x[n]$ and $x_d[n]$ are

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{T} - \frac{2\pi k}{T}\right)\right) , \\ X_d(e^{j\omega}) &= \frac{1}{MT} \sum_{r=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi r}{MT}\right)\right) . \end{aligned}$$

To relate $X(e^{j\omega})$ and $X_d(e^{j\omega})$, rewrite with

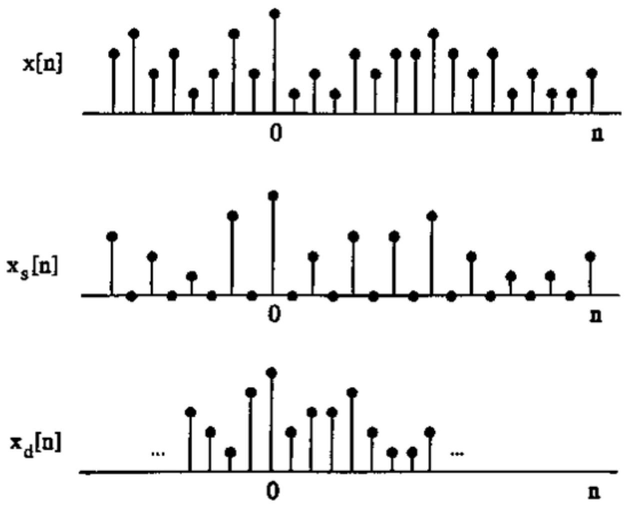
$$r = i + kM \qquad -\infty < k < \infty,\ 0 \le i \le M-1$$

$$\Longrightarrow X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1}\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\left(\frac{\omega}{MT} - \frac{2\pi k}{T} - \frac{2\pi i}{MT}\right)\right)\right]$$

$$= X(e^{j(\omega-2\pi i)/M})$$

$$\Longrightarrow X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1} X(e^{j(\omega-2\pi i)/M})\ .$$

As an example, the following figure illustrates decimation by $M = 2$ in the time domain. We see that re-sampling the continuous signal at MT is equivalent to keeping only every M-th sample. In this cascaded system, the value of T is arbitrary and not affected by the original sampling frequency of $x[n]$.



Time domain illustration of decimation at rate $M = 2$

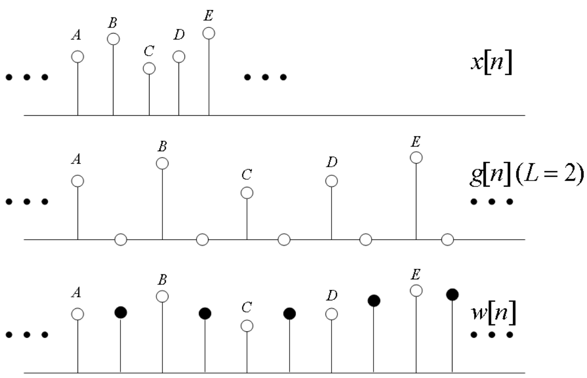
OSB Figure 4.21 shows the corresponding frequency-domain representation. In the frequency domain, a decimator can be viewed as a sequence of two operations: replication at $\frac{2\pi}{M}$, and frequency scaling by $\frac{1}{M}$. In general, the sampling rate of a signal can be reduced by a factor of M without aliasing if the signal is bandlimited to $\frac{\pi}{M}$. On the other hand, if the signal is not bandlimited, its bandwidth can be reduced first by discrete-time low pass filtering. Cascading an anti-aliasing filter with a decimator gives a downsampler. OSB Figure 4.22 illustrates downsampling with and without aliasing.

Sampling Rate Expansion by an Integer Factor

A typical system for increasing the sampling rate of a discrete sequence by an integer factor is illustrated in OSB Figure 4.24. Expressed in terms of Fourier transforms, the expander output is:

$$\begin{aligned} X_e(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_e[n]e^{-j\omega n} \\ &= \sum_{k=-\infty}^{\infty} x_e[k]e^{-j\omega kL} \quad = X(e^{j\omega L}) \end{aligned}$$

Expanding changes the time scale, and the LPF interpolates to fill in the missing values. As an example, the next figure shows upsampling at the rate of $L = 2$ in the time domain; for the corresponding spectra, see OSB Figure 4.25.



Time domain illustration of upsampling at rate $L = 2$

Changing the Sampling Rate by a Non-Integer Factor

By combining decimation and interpolation, the sampling rate of a sequence can be changed by a noninteger factor. For example, in OSB Figure 4.28 is a system for producing an output sequence with sampling period $\frac{TM}{L}$. It is preferred that the interpolator precedes the decimator to avoid possible aliasing, ie. decimation first may create aliasing since the spectrum is replicated at less than 2π. By comparison, when a compressor and an expander are cascaded (without the LPF's), it does not matter in what order they are placed, as long as the rates M and L are mutually prime.