LECTURE 12

LECTURE OUTLINE

- Convex Programming Duality
- Optimality Conditions
- Mixtures of Linear and Convex Constraints
- Existence of Optimal Primal Solutions
- Fenchel Duality
- Conic Duality

Reading: Sections 5.3.1-5.3.6

Line of analysis so far:

- Convex analysis (rel. int., dir. of recession, hyperplanes, conjugacy)
- MC/MC
- Nonlinear Farkas' Lemma
- Linear programming (duality, opt. conditions)
- We now discuss convex programming, and its many special cases (reliance on Nonlinear Farkas' Lemma)

CONVEX PROGRAMMING

Consider the problem

$$\begin{aligned} &\text{minimize} \ \ f(x) \\ &\text{subject to} \ \ x \in X, \ g_j(x) \le 0, \ j = 1, \ldots, r, \end{aligned}$$

where $X \subset \Re^n$ is convex, and $f : X \mapsto \Re$ and $g_j : X \mapsto \Re$ are convex. Assume f^*: finite.

- Recall the connection with the max crossing problem in the MC/MC framework where $M = \text{epi}(p)$ with

$$p(u) = \inf_{x \in X,\, g(x) \le u} f(x)$$

- Consider the Lagrangian function

$$L(x, \mu) = f(x) + \mu' g(x),$$

the dual function

$$q(\mu) = \begin{cases} \inf_{x \in X} L(x, \mu) & \text{if } \mu \ge 0, \\ -\infty & \text{otherwise} \end{cases}$$

and the dual problem of maximizing $\inf_{x \in X} L(x, \mu)$ over $\mu \ge 0$.

STRONG DUALITY THEOREM

- Assume that f^* is finite, and that one of the following two conditions holds:

  (1) There exists $x \in X$ such that $g(x) < 0$.

  (2) The functions g_j, $j = 1, \ldots, r$, are affine, and there exists $x \in \mathrm{ri}(X)$ such that $g(x) \leq 0$.

Then $q^* = f^*$ and the set of optimal solutions of the dual problem is nonempty. Under condition (1) this set is also compact.

- **Proof:** Replace $f(x)$ by $f(x) - f^*$ so that $f(x) - f^* \geq 0$ for all $x \in X$ w/ $g(x) \leq 0$. Apply Nonlinear Farkas' Lemma. Then, there exist $\mu_j^* \geq 0$, s.t.

$$f^* \leq f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x), \qquad \forall\, x \in X$$

- It follows that

$$f^* \leq \inf_{x \in X} \big\{ f(x) + \mu^{*\prime} g(x) \big\} \leq \inf_{x \in X,\, g(x) \leq 0} f(x) = f^*.$$

Thus equality holds throughout, and we have

$$f^* = \inf_{x \in X} \left\{ f(x) + \sum_{j=1}^{r} \mu_j^* g_j(x) \right\} = q(\mu^*)$$

QUADRATIC PROGRAMMING DUALITY

- Consider the quadratic program

$$\begin{aligned} &\text{minimize} \quad \tfrac{1}{2}x'Qx + c'x \\ &\text{subject to} \quad Ax \le b, \end{aligned}$$

where Q is positive definite.

- If f^* is finite, then $f^* = q^*$ and there exist both primal and dual optimal solutions, since the constraints are linear.

- Calculation of dual function:

$$q(\mu) = \inf_{x \in \Re^n} \left\{ \tfrac{1}{2}x'Qx + c'x + \mu'(Ax - b) \right\}$$

The infimum is attained for $x = -Q^{-1}(c + A'\mu)$, and, after substitution and calculation,

$$q(\mu) = -\tfrac{1}{2}\mu' AQ^{-1}A'\mu - \mu'(b + AQ^{-1}c) - \tfrac{1}{2}c'Q^{-1}c$$

- The dual problem, after a sign change, is

$$\begin{aligned} &\text{minimize} \quad \tfrac{1}{2}\mu'P\mu + t'\mu \\ &\text{subject to} \quad \mu \ge 0, \end{aligned}$$

where $P = AQ^{-1}A'$ and $t = b + AQ^{-1}c$.

OPTIMALITY CONDITIONS

- We have $q^* = f^*$, and the vectors x^* and μ^* are optimal solutions of the primal and dual problems, respectively, iff x^* is feasible, $\mu^* \geq 0$, and

$$x^* \in \arg\min_{x \in X} L(x, \mu^*), \qquad \mu_j^* g_j(x^*) = 0, \quad \forall\ j. \tag{1}$$

Proof: If $q^* = f^*$, and x^*, μ^* are optimal, then

$$\begin{aligned} f^* = q^* = q(\mu^*) = \inf_{x \in X} L(x, \mu^*) &\leq L(x^*, \mu^*) \\ &= f(x^*) + \sum_{j=1}^{r} \mu_j^* g_j(x^*) \leq f(x^*), \end{aligned}$$

where the last inequality follows from $\mu_j^* \geq 0$ and $g_j(x^*) \leq 0$ for all j. Hence equality holds throughout above, and (1) holds.

Conversely, if x^*, μ^* are feasible, and (1) holds,

$$\begin{aligned} q(\mu^*) = \inf_{x \in X} L(x, \mu^*) &= L(x^*, \mu^*) \\ &= f(x^*) + \sum_{j=1}^{r} \mu_j^* g_j(x^*) = f(x^*), \end{aligned}$$

so $q^* = f^*$, and x^*, μ^* are optimal. **Q.E.D.**

QUADRATIC PROGRAMMING OPT. COND.

For the quadratic program

$$\begin{aligned} &\text{minimize} \quad \tfrac{1}{2}x'Qx + c'x \\ &\text{subject to} \quad Ax \le b, \end{aligned}$$

where Q is positive definite, (x^*, μ^*) is a primal and dual optimal solution pair if and only if:

- Primal and dual feasibility holds:

$$Ax^* \le b, \qquad \mu^* \ge 0$$

- Lagrangian optimality holds [x^* minimizes $L(x, \mu^*)$ over $x \in \Re^n$]. This yields

$$x^* = -Q^{-1}(c + A'\mu^*)$$

- Complementary slackness holds [$(Ax^* - b)'\mu^* = 0$]. It can be written as

$$\mu_j^* > 0 \qquad \Rightarrow \qquad a_j'x^* = b_j, \quad \forall\, j = 1, \ldots, r,$$

where a_j' is the jth row of A, and b_j is the jth component of b.

LINEAR EQUALITY CONSTRAINTS

- The problem is

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in X, \quad g(x) \leq 0, \quad Ax = b, \end{array}$$

where X is convex, $g(x) = \big(g_1(x), \ldots, g_r(x)\big)'$, $f : X \mapsto \Re$ and $g_j : X \mapsto \Re$, $j = 1, \ldots, r$, are convex.

- Convert the constraint $Ax = b$ to $Ax \leq b$ and $-Ax \leq -b$, with corresponding dual variables $\lambda^+ \geq 0$ and $\lambda^- \geq 0$.

- The Lagrangian function is

$$f(x) + \mu' g(x) + (\lambda^+ - \lambda^-)'(Ax - b),$$

and by introducing a dual variable $\lambda = \lambda^+ - \lambda^-$, with no sign restriction, it can be written as

$$L(x, \mu, \lambda) = f(x) + \mu' g(x) + \lambda'(Ax - b).$$

- The dual problem is

$$\begin{array}{ll} \text{maximize} & q(\mu, \lambda) \equiv \inf_{x \in X} L(x, \mu, \lambda) \\ \text{subject to} & \mu \geq 0, \ \lambda \in \Re^m. \end{array}$$

DUALITY AND OPTIMALITY COND.

- **Pure equality constraints:**

  (a) Assume that f^*: finite and there exists $x \in \mathrm{ri}(X)$ such that $Ax = b$. Then $f^* = q^*$ and there exists a dual optimal solution.

  (b) $f^* = q^*$, and (x^*, λ^*) are a primal and dual optimal solution pair if and only if x^* is feasible, and

$$x^* \in \arg\min_{x \in X} L(x, \lambda^*)$$

Note: No complementary slackness for equality constraints.

- **Linear and nonlinear constraints:**

  (a) Assume f^*: finite, that there exists $x \in X$ such that $Ax = b$ and $g(x) < 0$, and that there exists $\tilde{x} \in \mathrm{ri}(X)$ such that $A\tilde{x} = b$. Then $q^* = f^*$ and there exists a dual optimal solution.

  (b) $f^* = q^*$, and (x^*, μ^*, λ^*) are a primal and dual optimal solution pair if and only if x^* is feasible, $\mu^* \geq 0$, and

$$x^* \in \arg\min_{x \in X} L(x, \mu^*, \lambda^*), \quad \mu_j^* g_j(x^*) = 0, \quad \forall\, j$$

COUNTEREXAMPLE I

- **Strong Duality Counterexample:** Consider

$$\begin{aligned}&\text{minimize} \quad f(x) = e^{-\sqrt{x_1 x_2}}\\ &\text{subject to} \quad x_1 = 0, \qquad x \in X = \{x \mid x \geq 0\}\end{aligned}$$

Here $f^* = 1$ and f is convex (its Hessian is > 0 in the interior of X). The dual function is

$$q(\lambda) = \inf_{x \geq 0} \left\{ e^{-\sqrt{x_1 x_2}} + \lambda x_1 \right\} = \begin{cases} 0 & \text{if } \lambda \geq 0, \\ -\infty & \text{otherwise,} \end{cases}$$

(when $\lambda \geq 0$, the expression in braces is nonnegative for $x \geq 0$ and can approach zero by taking $x_1 \to 0$ and $x_1 x_2 \to \infty$). Thus $q^* = 0$.

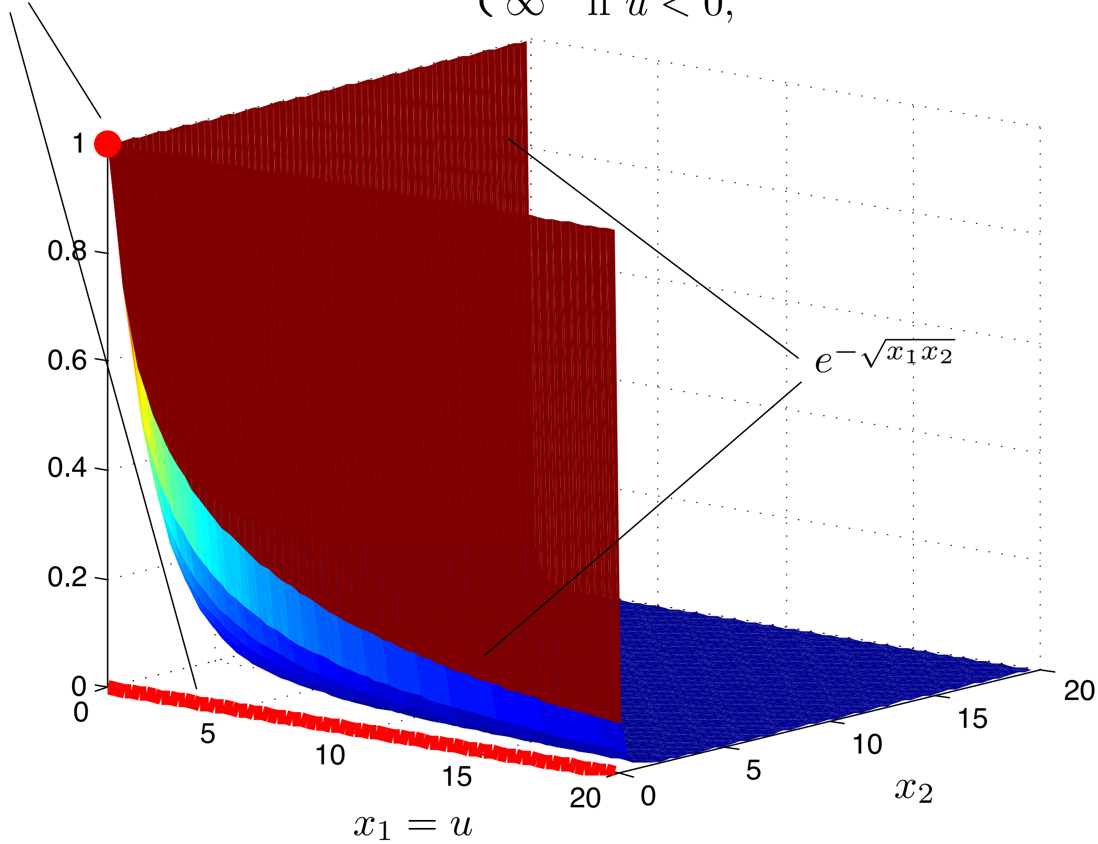
- The relative interior assumption is violated.
- As predicted by the corresponding MC/MC framework, the perturbation function

$$p(u) = \inf_{x_1 = u,\, x \geq 0} e^{-\sqrt{x_1 x_2}} = \begin{cases} 0 & \text{if } u > 0, \\ 1 & \text{if } u = 0, \\ \infty & \text{if } u < 0, \end{cases}$$

is not lower semicontinuous at $u = 0$.

COUNTEREXAMPLE VISUALIZATION

$$p(u) = \inf_{x_1 = u,\, x \geq 0} e^{-\sqrt{x_1 x_2}} = \begin{cases} 0 & \text{if } u > 0, \\ 1 & \text{if } u = 0, \\ \infty & \text{if } u < 0, \end{cases}$$



- Connection with counterexample for preservation of closedness under partial minimization.

COUNTEREXAMPLE II

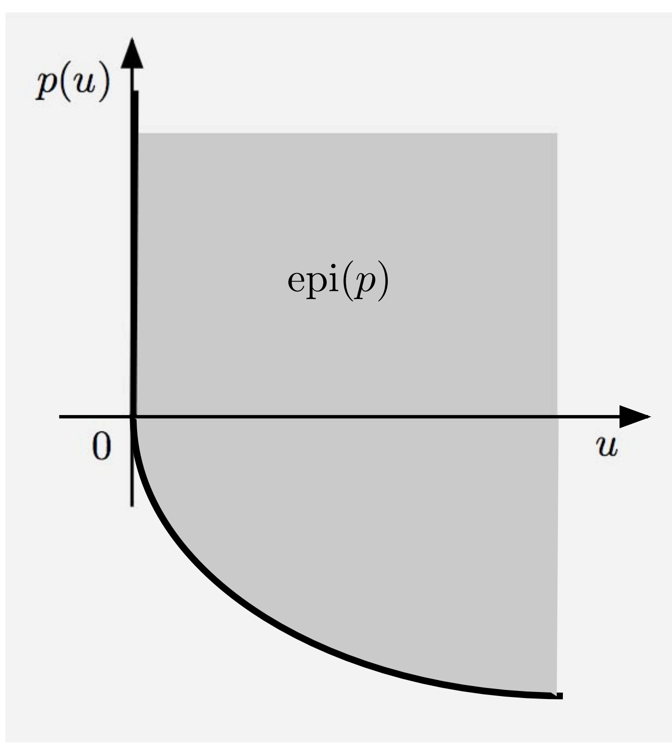
- **Existence of Solutions Counterexample:** Let $X = \Re$, $f(x) = x$, $g(x) = x^2$. Then $x^* = 0$ is the only feasible/optimal solution, and we have

$$q(\mu) = \inf_{x \in \Re} \{x + \mu x^2\} = -\frac{1}{4\mu}, \qquad \forall\, \mu > 0,$$

and $q(\mu) = -\infty$ for $\mu \leq 0$, so that $q^* = f^* = 0$. However, there is no $\mu^* \geq 0$ such that $q(\mu^*) = q^* = 0$.

- The perturbation function is

$$p(u) = \inf_{x^2 \leq u} x = \begin{cases} -\sqrt{u} & \text{if } u \geq 0, \\ \infty & \text{if } u < 0. \end{cases}$$



FENCHEL DUALITY FRAMEWORK

- Consider the problem

$$\begin{aligned} &\text{minimize} \quad f_1(x) + f_2(x) \\ &\text{subject to} \ \ x \in \Re^n, \end{aligned}$$

where $f_1 : \Re^n \mapsto (-\infty, \infty]$ and $f_2 : \Re^n \mapsto (-\infty, \infty]$ are closed proper convex functions.

- Convert to the equivalent problem

$$\begin{aligned} &\text{minimize} \quad f_1(x_1) + f_2(x_2) \\ &\text{subject to} \ \ x_1 = x_2, \quad x_1 \in \mathrm{dom}(f_1), \ x_2 \in \mathrm{dom}(f_2) \end{aligned}$$

- The dual function is

$$\begin{aligned} q(\lambda) &= \inf_{x_1 \in \mathrm{dom}(f_1), \ x_2 \in \mathrm{dom}(f_2)} \big\{ f_1(x_1) + f_2(x_2) + \lambda'(x_2 - x_1) \\ &= \inf_{x_1 \in \Re^n} \big\{ f_1(x_1) - \lambda' x_1 \big\} + \inf_{x_2 \in \Re^n} \big\{ f_2(x_2) + \lambda' x_2 \big\} \end{aligned}$$

- **Dual problem:** $\max_\lambda \{-f_1^\star(\lambda) - f_2^\star(-\lambda)\} = -\min_\lambda \{-q(\lambda)\}$ or

$$\begin{aligned} &\text{minimize} \quad f_1^\star(\lambda) + f_2^\star(-\lambda) \\ &\text{subject to} \ \ \lambda \in \Re^n, \end{aligned}$$

where $f_1^\star$ and $f_2^\star$ are the conjugates.

FENCHEL DUALITY THEOREM

- Consider the Fenchel framework:

  (a) If f^* is finite and $\mathrm{ri}\big(\mathrm{dom}(f_1)\big) \cap \mathrm{ri}\big(\mathrm{dom}(f_2)\big) \neq \emptyset$, then $f^* = q^*$ and there exists at least one dual optimal solution.

  (b) There holds $f^* = q^*$, and (x^*, λ^*) is a primal and dual optimal solution pair if and only if

$$x^* \in \arg \min_{x \in \Re^n} \big\{ f_1(x) - x'\lambda^* \big\}, \ \ x^* \in \arg \min_{x \in \Re^n} \big\{ f_2(x) + x'\lambda^* \big\}$$

Proof: For strong duality use the equality constrained problem

$$\begin{aligned} &\text{minimize} \quad f_1(x_1) + f_2(x_2) \\ &\text{subject to} \ \ x_1 = x_2, \quad x_1 \in \mathrm{dom}(f_1), \ x_2 \in \mathrm{dom}(f_2) \end{aligned}$$

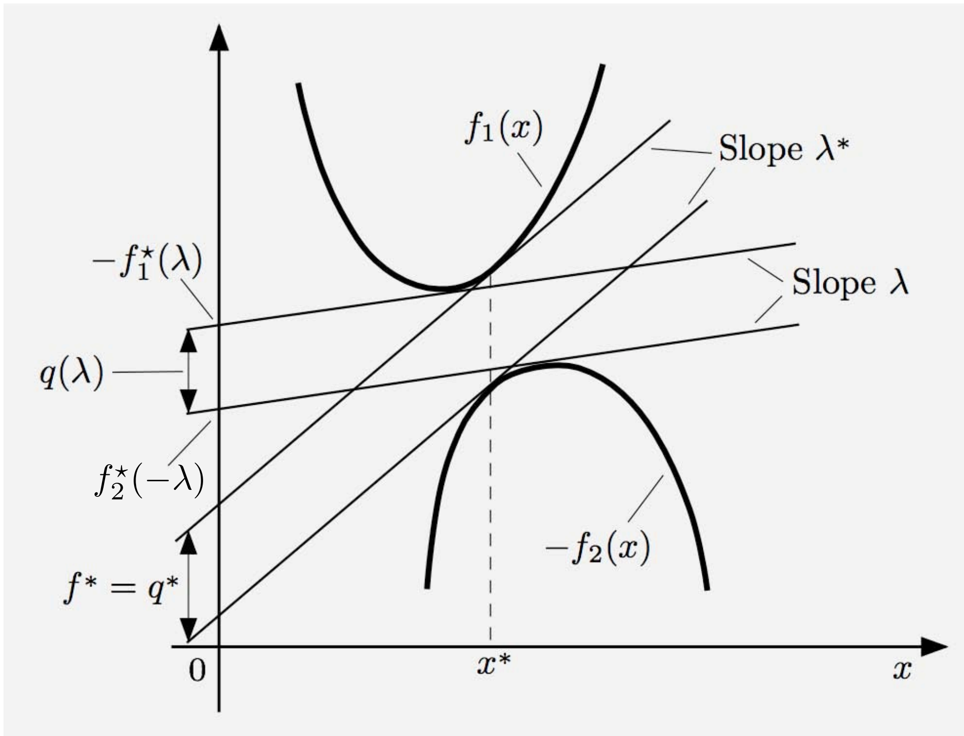
and the fact

$$\mathrm{ri}\big(\mathrm{dom}(f_1) \times \mathrm{dom}(f_2)\big) = \mathrm{ri}\big(\mathrm{dom}(f_1)\big) \times \big(\mathrm{dom}(f_2)\big)$$

to satisfy the relative interior condition.

For part (b), apply the optimality conditions (primal and dual feasibility, and Lagrangian optimality).

GEOMETRIC INTERPRETATION



- When $\mathrm{dom}(f_1) = \mathrm{dom}(f_2) = \Re^n$, and f_1 and f_2 are differentiable, the optimality condition is equivalent to

$$\lambda^* = \nabla f_1(x^*) = -\nabla f_2(x^*)$$

- By reversing the roles of the (symmetric) primal and dual problems, we obtain alternative criteria for strong duality: if q^* is finite and $\mathrm{ri}\big(\mathrm{dom}(f_1^\star)\big) \cap \mathrm{ri}\big(-\mathrm{dom}(f_2^\star)\big) \neq \emptyset$, then $f^* = q^*$ and there exists at least one primal optimal solution.

CONIC PROBLEMS

- A conic problem is to minimize a convex function $f : \Re^n \mapsto (-\infty, \infty]$ subject to a cone constraint.

- The most useful/popular special cases:
  - Linear-conic programming
  - Second order cone programming
  - Semidefinite programming

involve minimization of a linear function over the intersection of an affine set and a cone.

- Can be analyzed as a special case of Fenchel duality.

- There are many interesting applications of conic problems, including in discrete optimization.

CONIC DUALITY

- Consider minimizing $f(x)$ over $x \in C$, where $f : \Re^n \mapsto (-\infty, \infty]$ is a closed proper convex function and C is a closed convex cone in $\Re^n$.

- We apply Fenchel duality with the definitions

$$f_1(x) = f(x), \qquad f_2(x) = \begin{cases} 0 & \text{if } x \in C, \\ \infty & \text{if } x \notin C. \end{cases}$$

The conjugates are

$$f_1^\star(\lambda) = \sup_{x \in \Re^n} \big\{\lambda' x - f(x)\big\}, \; f_2^\star(\lambda) = \sup_{x \in C} \lambda' x = \begin{cases} 0 & \text{if } \lambda \in C^*, \\ \infty & \text{if } \lambda \notin C^*, \end{cases}$$

where $C^* = \{\lambda \mid \lambda' x \le 0, \forall\, x \in C\}$.

- The dual problem is

$$\begin{aligned} &\text{minimize} \quad f^\star(\lambda) \\ &\text{subject to} \;\; \lambda \in \hat{C}, \end{aligned}$$

where $f^\star$ is the conjugate of f and

$$\hat{C} = \{\lambda \mid \lambda' x \ge 0, \forall\, x \in C\}.$$

$\hat{C}$ and $-\hat{C}$ are called the *dual* and *polar* cones.

CONIC DUALITY THEOREM

- Assume that the optimal value of the primal conic problem is finite, and that

$$\mathrm{ri}\big(\mathrm{dom}(f)\big) \cap \mathrm{ri}(C) \neq \emptyset.$$

Then, there is no duality gap and the dual problem has an optimal solution.

- Using the symmetry of the primal and dual problems, we also obtain that there is no duality gap and the primal problem has an optimal solution if the optimal value of the dual conic problem is finite, and

$$\mathrm{ri}\big(\mathrm{dom}(f^\star)\big) \cap \mathrm{ri}(\hat{C}) \neq \emptyset.$$

LINEAR CONIC PROGRAMMING

- Let f be linear over its domain, i.e.,

$$f(x) = \begin{cases} c'x & \text{if } x \in X, \\ \infty & \text{if } x \notin X, \end{cases}$$

where c is a vector, and $X = b + S$ is an affine set.

- Primal problem is

$$\begin{aligned} &\text{minimize} \quad c'x \\ &\text{subject to} \;\; x - b \in S, \quad x \in C. \end{aligned}$$

- We have

$$\begin{aligned} f^\star(\lambda) &= \sup_{x-b\in S} (\lambda - c)'x = \sup_{y \in S} (\lambda - c)'(y + b) \\ &= \begin{cases} (\lambda - c)'b & \text{if } \lambda - c \in S^\perp, \\ \infty & \text{if } \lambda - c \notin S. \end{cases} \end{aligned}$$

- Dual problem is equivalent to

$$\begin{aligned} &\text{minimize} \quad b'\lambda \\ &\text{subject to} \;\; \lambda - c \in S^\perp, \quad \lambda \in \hat{C}. \end{aligned}$$

- If $X \cap \mathrm{ri}(C) = \emptyset$, there is no duality gap and there exists a dual optimal solution.

ANOTHER APPROACH TO DUALITY

- Consider the problem

$$\begin{aligned} &\text{minimize} \quad f(x) \\ &\text{subject to} \quad x \in X, \; g_j(x) \leq 0, \; j = 1, \ldots, r \end{aligned}$$

and perturbation fn $p(u) = \inf_{x \in X,\, g(x) \leq u} f(x)$

- Recall the MC/MC framework with $M = \text{epi}(p)$. Assuming that p is convex and $f^* < \infty$, by 1st MC/MC theorem, we have $f^* = q^*$ if and only if p is lower semicontinuous at 0.

- **Duality Theorem:** Assume that X, f, and g_j are closed convex, and the feasible set is nonempty and compact. Then $f^* = q^*$ and the set of optimal primal solutions is nonempty and compact.

Proof: Use partial minimization theory w/ the function

$$F(x,u) = \begin{cases} f(x) & \text{if } x \in X,\, g(x) \leq u, \\ \infty & \text{otherwise.} \end{cases}$$

p is obtained by the partial minimization:

$$p(u) = \inf_{x \in \Re^n} F(x,u).$$

Under the given assumption, p is closed convex.

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