LECTURE 8

LECTURE OUTLINE

- Convex conjugate functions
- Conjugacy theorem
- Examples
- Support functions

Reading: Section 1.6

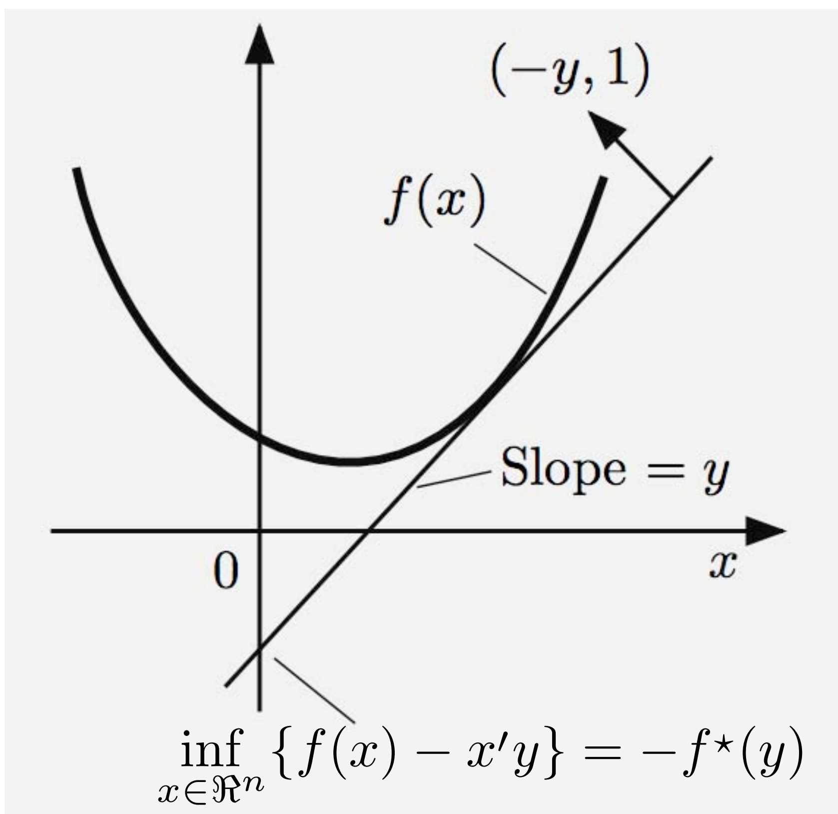
CONJUGATE CONVEX FUNCTIONS

- Consider a function f and its epigraph

Nonvertical hyperplanes supporting epi(f)

$\mapsto$ Crossing points of vertical axis

$$f^\star(y) = \sup_{x \in \Re^n} \big\{x'y - f(x)\big\}, \qquad y \in \Re^n.$$

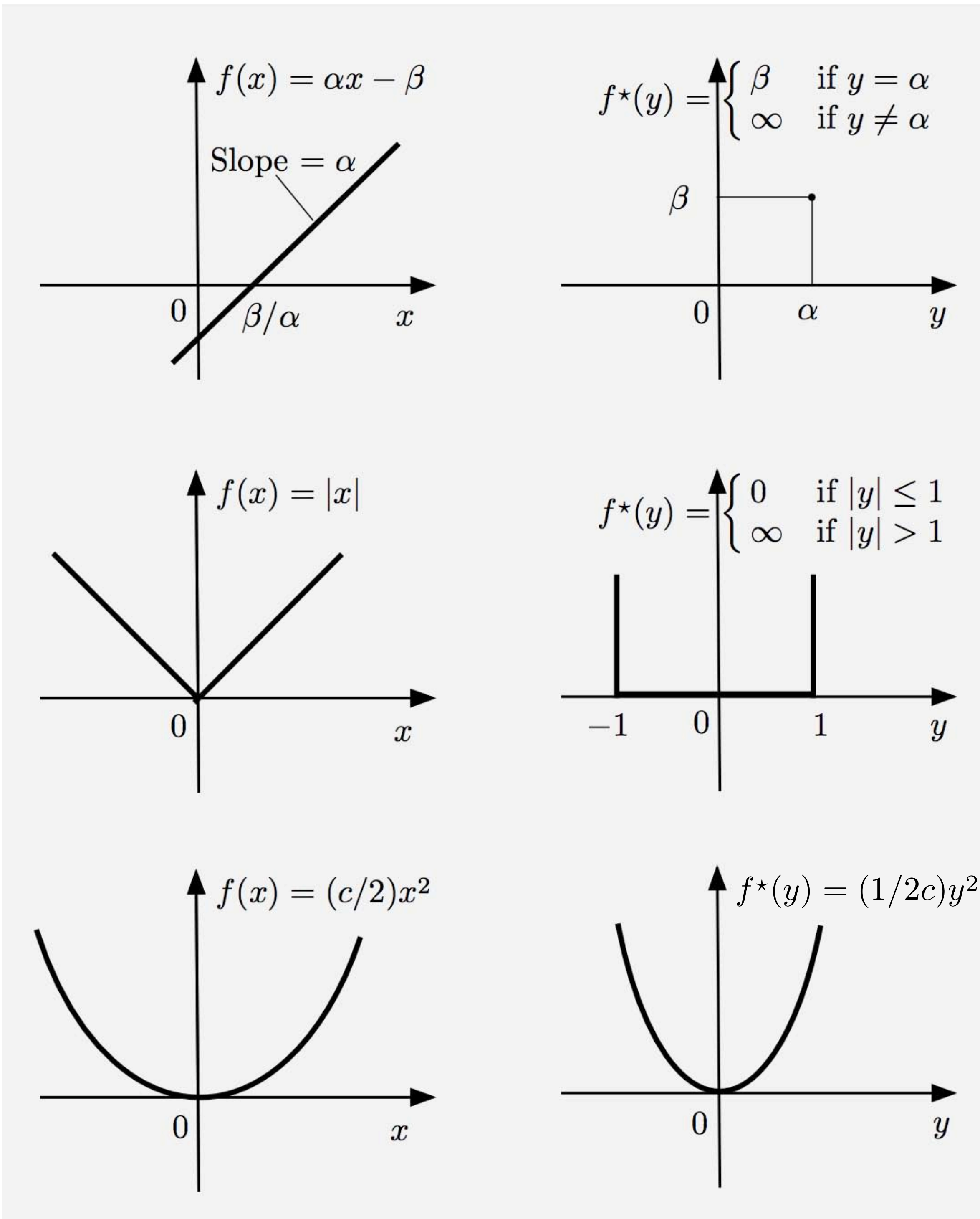


- For any $f : \Re^n \mapsto [-\infty, \infty]$, its *conjugate convex function* is defined by

$$f^\star(y) = \sup_{x \in \Re^n} \big\{x'y - f(x)\big\}, \qquad y \in \Re^n$$

EXAMPLES

$$f^\star(y) = \sup_{x\in\Re^n}\big\{x'y - f(x)\big\}, \qquad y \in \Re^n$$



CONJUGATE OF CONJUGATE

- From the definition

$$f^\star(y) = \sup_{x \in \Re^n} \big\{x'y - f(x)\big\}, \qquad y \in \Re^n,$$

note that $f^\star$ *is convex and closed.*

- Reason: $\text{epi}(f^\star)$ is the intersection of the epigraphs of the linear functions of y

$$x'y - f(x)$$

as x ranges over $\Re^n$.

- Consider the conjugate of the conjugate:

$$f^{\star\star}(x) = \sup_{y \in \Re^n} \big\{y'x - f^\star(y)\big\}, \qquad x \in \Re^n.$$

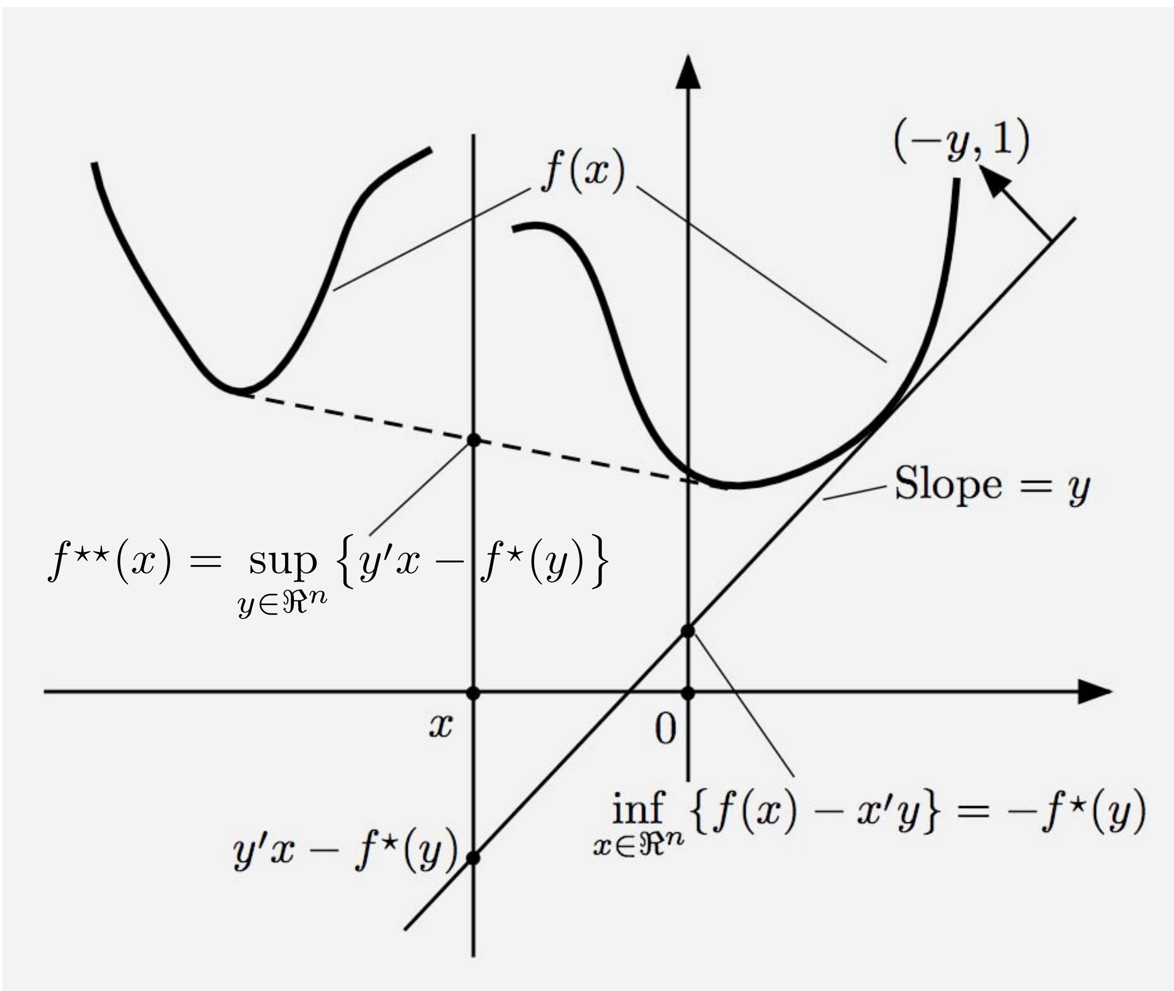
- $f^{\star\star}$ is convex and closed.
- **Important fact/Conjugacy theorem:** If f is closed proper convex, then $f^{\star\star} = f$.

CONJUGACY THEOREM - VISUALIZATION

$$f^\star(y) = \sup_{x\in\Re^n}\big\{x'y - f(x)\big\}, \qquad y \in \Re^n$$

$$f^{\star\star}(x) = \sup_{y\in\Re^n}\big\{y'x - f^\star(y)\big\}, \qquad x \in \Re^n$$

- If f is closed convex proper, then $f^{\star\star} = f$.



CONJUGACY THEOREM

- Let $f : \Re^n \mapsto (-\infty, \infty]$ be a function, let $\check{\mathrm{cl}}\, f$ be its convex closure, let $f^\star$ be its convex conjugate, and consider the conjugate of $f^\star$,

$$f^{\star\star}(x) = \sup_{y \in \Re^n} \big\{ y'x - f^\star(y) \big\}, \qquad x \in \Re^n$$

(a) We have

$$f(x) \geq f^{\star\star}(x), \qquad \forall\, x \in \Re^n$$

(b) If f is convex, then properness of any one of f, $f^\star$, and $f^{\star\star}$ implies properness of the other two.

(c) If f is closed proper and convex, then

$$f(x) = f^{\star\star}(x), \qquad \forall\, x \in \Re^n$$

(d) If $\check{\mathrm{cl}}\, f(x) > -\infty$ for all $x \in \Re^n$, then

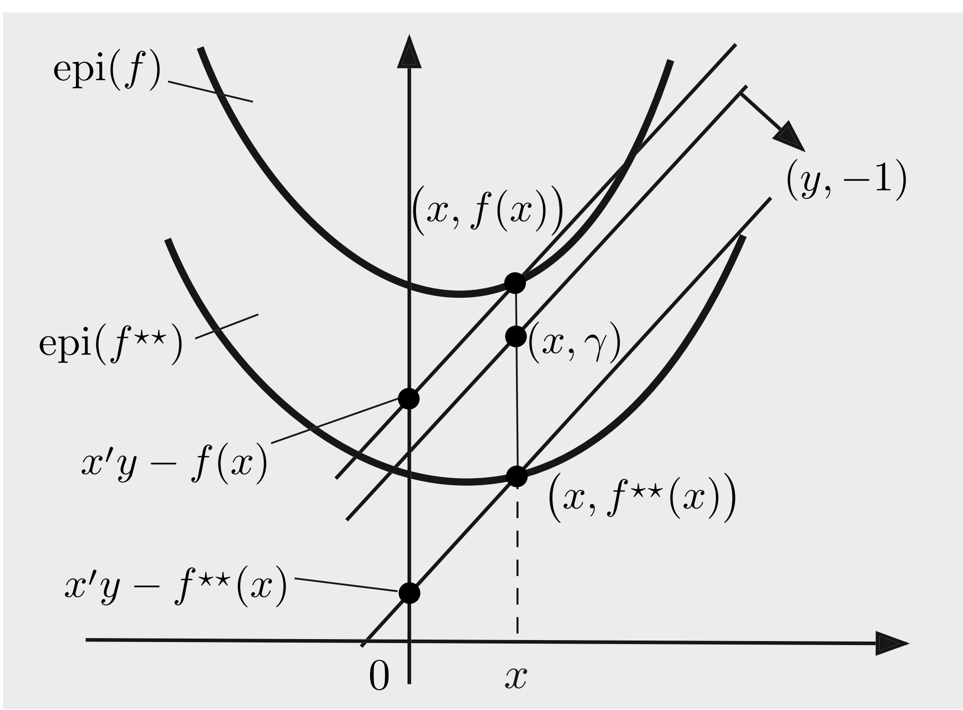
$$\check{\mathrm{cl}}\, f(x) = f^{\star\star}(x), \qquad \forall\, x \in \Re^n$$

PROOF OF CONJUGACY THEOREM (A), (C)

- (a) For all x, y, we have $f^\star(y) \geq y'x - f(x)$, implying that $f(x) \geq \sup_y \{y'x - f^\star(y)\} = f^{\star\star}(x)$.
- (c) By contradiction. Assume there is $(x,\gamma) \in \text{epi}(f^{\star\star})$ with $(x,\gamma) \notin \text{epi}(f)$. There exists a nonvertical hyperplane with normal $(y, -1)$ that strictly separates (x,γ) and $\text{epi}(f)$. (The vertical component of the normal vector is normalized to -1.)
- Consider two parallel hyperplanes, translated to pass through $\big(x, f(x)\big)$ and $\big(x, f^{\star\star}(x)\big)$. Their vertical crossing points are $x'y - f(x)$ and $x'y - f^{\star\star}(x)$, and lie strictly above and below the crossing point of the strictly sep. hyperplane. Hence

$$x'y - f(x) > x'y - f^{\star\star}(x)$$

which contradicts part (a). **Q.E.D.**



A COUNTEREXAMPLE

- A counterexample (with closed convex but improper f) showing the need to assume properness in order for $f = f^{\star\star}$:

$$f(x) = \begin{cases} \infty & \text{if } x > 0, \\ -\infty & \text{if } x \le 0. \end{cases}$$

We have

$$f^{\star}(y) = \infty, \qquad \forall\, y \in \Re^n,$$

$$f^{\star\star}(x) = -\infty, \qquad \forall\, x \in \Re^n.$$

But

$$\check{\mathrm{cl}}\, f = f,$$

so $\check{\mathrm{cl}}\, f \neq f^{\star\star}$.

A FEW EXAMPLES

- l_p and l_q norm conjugacy, where $\frac{1}{p} + \frac{1}{q} = 1$

$$f(x) = \frac{1}{p} \sum_{i=1}^{n} |x_i|^p, \qquad f^\star(y) = \frac{1}{q} \sum_{i=1}^{n} |y_i|^q$$

- Conjugate of a strictly convex quadratic

$$f(x) = \frac{1}{2} x' Q x + a' x + b,$$

$$f^\star(y) = \frac{1}{2} (y - a)' Q^{-1} (y - a) - b.$$

- Conjugate of a function obtained by invertible linear transformation/translation of a function p

$$f(x) = p\big(A(x - c)\big) + a' x + b,$$

$$f^\star(y) = q\big((A')^{-1}(y - a)\big) + c' y + d,$$

where q is the conjugate of p and $d = -(c'a + b)$.

SUPPORT FUNCTIONS

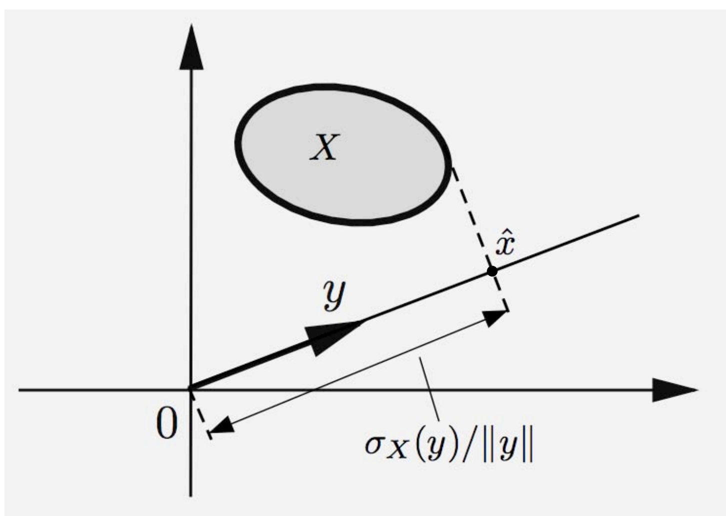
- Conjugate of indicator function δ_X of set X

$$\sigma_X(y) = \sup_{x \in X} y'x$$

is called the *support function of* X.

- To determine $\sigma_X(y)$ for a given vector y, we project the set X on the line determined by y, we find $\hat{x}$, the extreme point of projection in the direction y, and we scale by setting

$$\sigma_X(y) = \|\hat{x}\| \cdot \|y\|$$



- epi(σ_X) is a closed convex cone.

- The sets X, cl(X), conv(X), and cl(conv(X)) all have the same support function (by the conjugacy theorem).

SUPPORT FN OF A CONE - POLAR CONE

- The conjugate of the indicator function δ_C is the support function, $\sigma_C(y) = \sup_{x\in C} y'x$.

- If C is a cone,

$$\sigma_C(y) = \begin{cases} 0 & \text{if } y'x \le 0, \ \forall\, x \in C, \\ \infty & \text{otherwise} \end{cases}$$

i.e., σ_C is the indicator function δ_{C^*} of the cone

$$C^* = \{y \mid y'x \le 0, \ \forall\, x \in C\}$$

This is called the *polar cone of* C.

- By the Conjugacy Theorem the polar cone of C^* is $\mathrm{cl}\big(\mathrm{conv}(C)\big)$. This is the *Polar Cone Theorem.*

- **Special case:** If $C = \mathrm{cone}\big(\{a_1, \ldots, a_r\}\big)$, then

$$C^* = \{x \mid a_j'x \le 0, \ j = 1, \ldots, r\}$$

- **Farkas' Lemma:** $(C^*)^* = C$.

- True because C is a closed set $[\mathrm{cone}\big(\{a_1, \ldots, a_r\}\big)$ is the image of the positive orthant $\{\alpha \mid \alpha \ge 0\}$ under the linear transformation that maps α to $\sum_{j=1}^r \alpha_j a_j]$, and the image of any polyhedral set under a linear transformation is a closed set.

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