LECTURE 6

LECTURE OUTLINE

- Nonemptiness of closed set intersections
- Existence of optimal solutions
- Linear and quadratic programming
- Preservation of closure under linear transformation

ROLE OF CLOSED SET INTERSECTIONS I

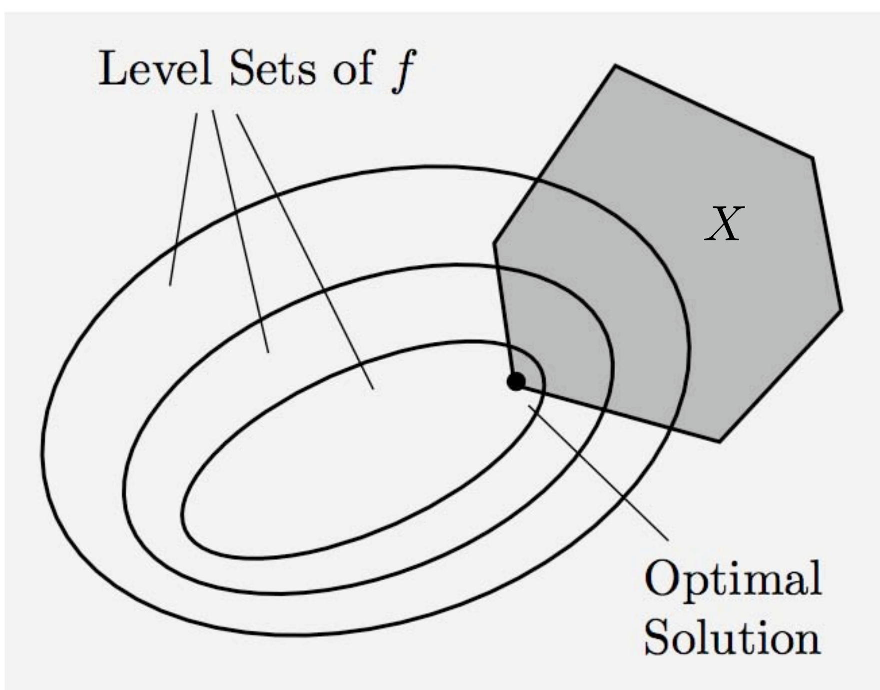
- **A fundamental question:** Given a sequence of nonempty closed sets $\{C_k\}$ in $\Re^n$ with $C_{k+1} \subset C_k$ for all k, when is $\cap_{k=0}^{\infty} C_k$ nonempty?

- Set intersection theorems are significant in at least three major contexts, which we will discuss in what follows:

1. Does a function $f : \Re^n \mapsto (-\infty, \infty]$ attain a minimum over a set X? This is true if and only if

$$\text{Intersection of nonempty } \big\{x \in X \mid f(x) \leq \gamma_k\big\}$$

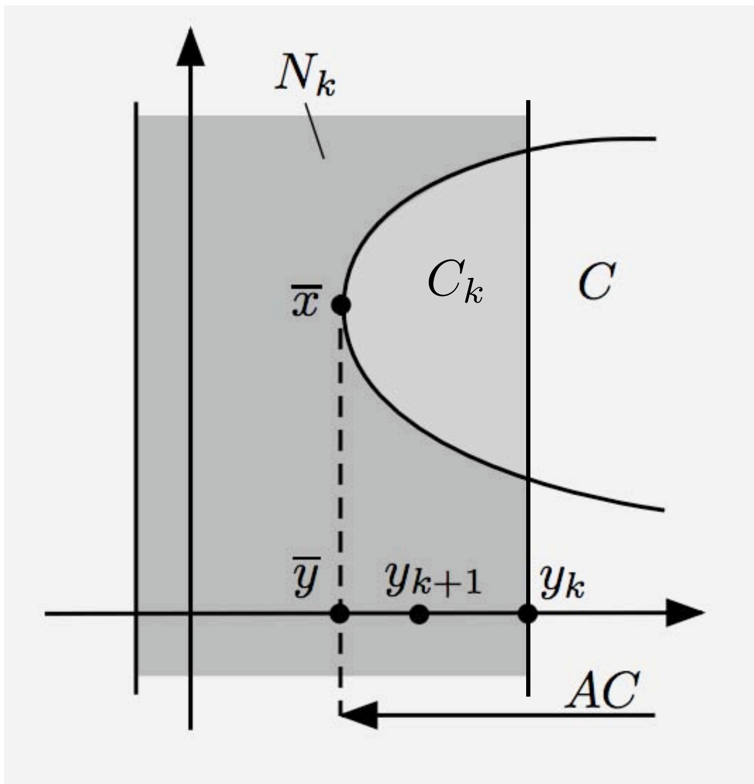
is nonempty.



ROLE OF CLOSED SET INTERSECTIONS II

2. If C is closed and A is a matrix, is AC closed? Special case:

- If C_1 and C_2 are closed, is $C_1 + C_2$ closed?



3. If $F(x, z)$ is closed, is $f(x) = \inf_z F(x, z)$ closed? (Critical question in duality theory.) Can be addressed by using the relation

$$P\big(\text{epi}(F)\big) \subset \text{epi}(f) \subset \text{cl}\Big(P\big(\text{epi}(F)\big)\Big)$$

where $P(\cdot)$ is projection on the space of (x, w).

ASYMPTOTIC SEQUENCES

- Given nested sequence $\{C_k\}$ of closed convex sets, $\{x_k\}$ is an *asymptotic sequence* if

$$x_k \in C_k, \qquad x_k \neq 0, \qquad k = 0, 1, \ldots$$

$$\|x_k\| \to \infty, \qquad \frac{x_k}{\|x_k\|} \to \frac{d}{\|d\|}$$

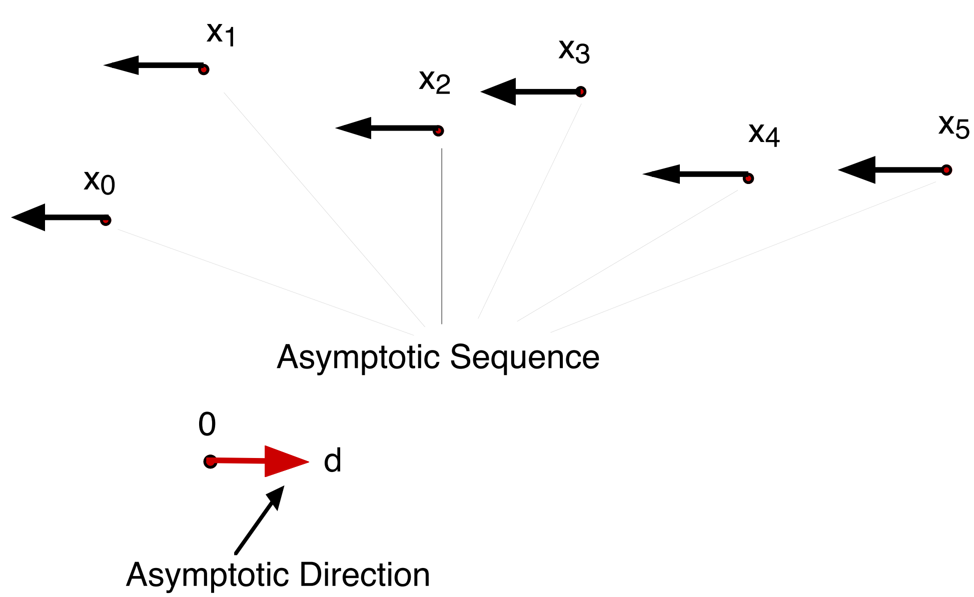
where d is a nonzero common direction of recession of the sets C_k.

- As a special case we define asymptotic sequence of a closed convex set C (use $C_k \equiv C$).

- Every unbounded $\{x_k\}$ with $x_k \in C_k$ has an asymptotic subsequence.

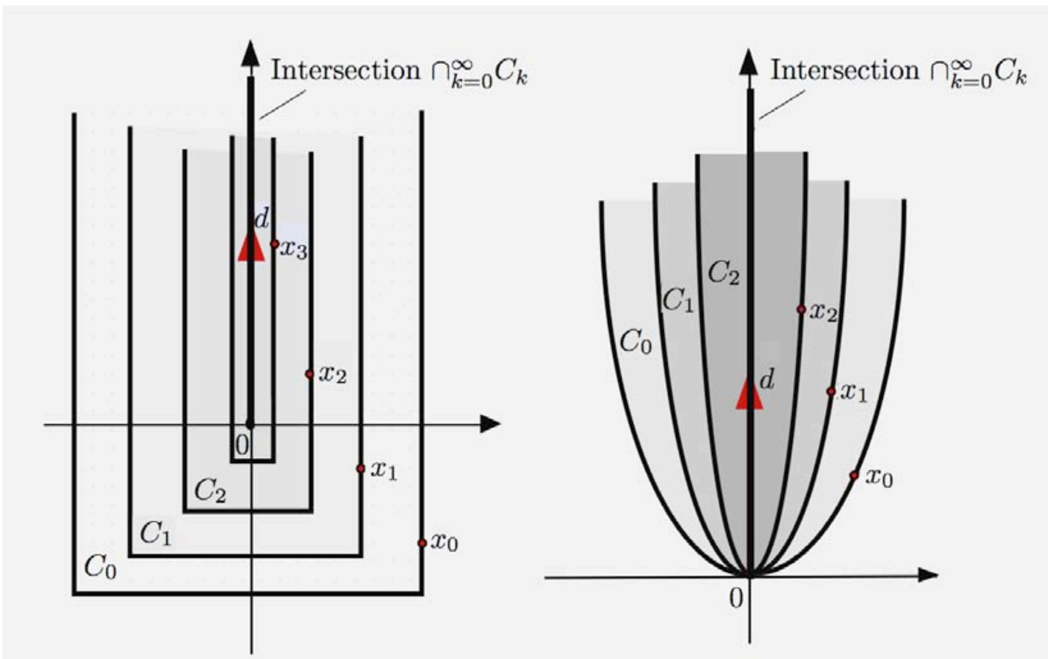
- $\{x_k\}$ is called *retractive* if for some k, we have

$$x_k - d \in C_k, \quad \forall\, k \geq k.$$



RETRACTIVE SEQUENCES

- A nested sequence $\{C_k\}$ of closed convex sets is *retractive* if all its asymptotic sequences are retractive.



(a) Retractive Set Sequence (b) Nonretractive Set Sequence

- A closed halfspace (viewed as a sequence with identical components) is retractive.

- Intersections and Cartesian products of retractive set sequences are retractive.

- A polyhedral set is retractive. Also the vector sum of a convex compact set and a retractive convex set is retractive.

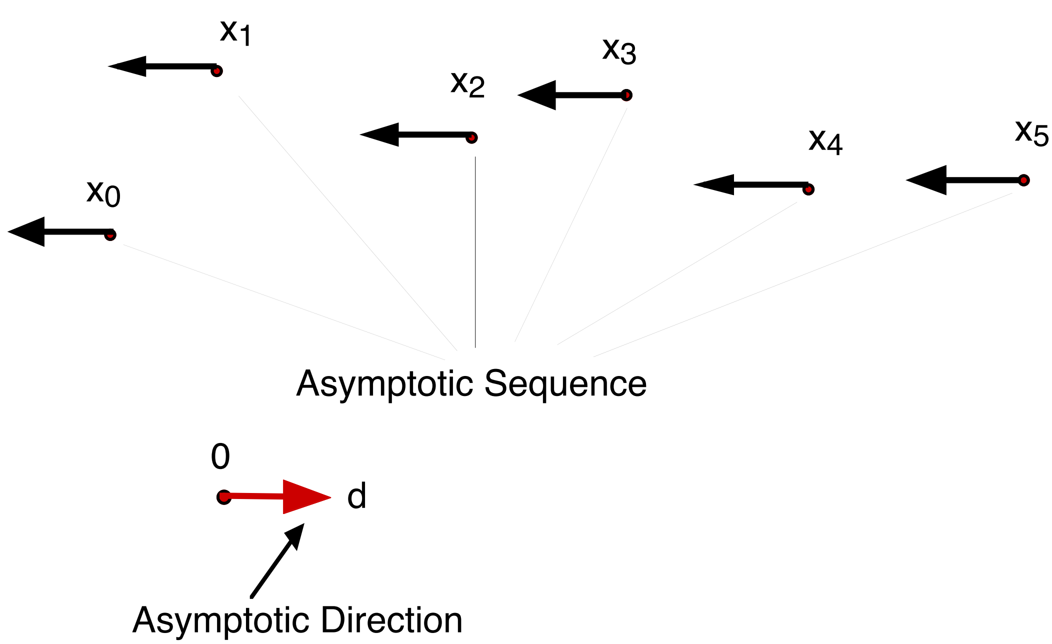
- Nonpolyhedral cones and level sets of quadratic functions need not be retractive.

SET INTERSECTION THEOREM I

Proposition: If $\{C_k\}$ is retractive, then $\cap_{k=0}^{\infty} C_k$ is nonempty.

- Key proof ideas:
  (a) The intersection $\cap_{k=0}^{\infty} C_k$ is empty iff the sequence $\{x_k\}$ of minimum norm vectors of C_k is unbounded (so a subsequence is asymptotic).

  (b) An asymptotic sequence $\{x_k\}$ of minimum norm vectors cannot be retractive, because such a sequence eventually gets closer to 0 when shifted opposite to the asymptotic direction.



SET INTERSECTION THEOREM II

Proposition: Let $\{C_k\}$ be a nested sequence of nonempty closed convex sets, and X be a retractive set such that all the sets $C_k = X \cap C_k$ are nonempty. Assume that

$$R_X \cap R \subset L,$$

where

$$R = \cap_{k=0}^{\infty} R_{C_k}, \qquad L = \cap_{k=0}^{\infty} L_{C_k}$$

Then $\{C_k\}$ is retractive and $\cap_{k=0}^{\infty} C_k$ is nonempty.

- Special cases:
  - $X = \Re^n$, $R = L$ ("cylindrical" sets C_k)
  - $R_X \cap R = \{0\}$ (no nonzero common recession direction of X and $\cap_k C_k$)

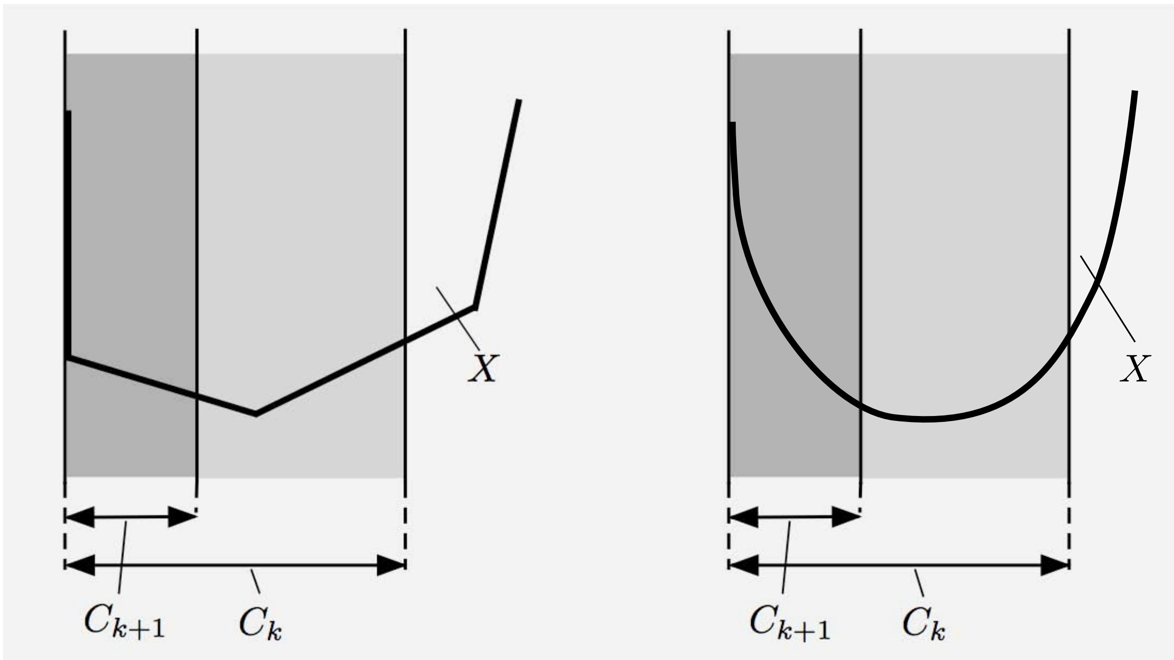
Proof: The set of common directions of recession of C_k is $R_X \cap R$. For any asymptotic sequence $\{x_k\}$ corresponding to $d \in R_X \cap R$:

(1) $x_k - d \in C_k$ (because $d \in L$)

(2) $x_k - d \in X$ (because X is retractive)

So $\{C_k\}$ is retractive.

NEED TO ASSUME THAT X IS RETRACTIVE



Consider $\cap_{k=0}^{\infty} \bar{C}_k$, with $\bar{C}_k = X \cap C_k$

- The condition $R_X \cap R \subset L$ holds
- In the figure on the left, X is polyhedral.
- In the figure on the right, X is nonpolyhedral and nonretrative, and

$$\cap_{k=0}^{\infty} \bar{C}_k = \emptyset$$

LINEAR AND QUADRATIC PROGRAMMING

- **Theorem:** Let

$$f(x) = x'Qx + c'x, \quad X = \{x \mid a_j'x + b_j \le 0,\ j = 1, \ldots, r\}$$

where Q is symmetric positive semidefinite. If the minimal value of f over X is finite, there exists a minimum of f over X.

Proof: (Outline) Write

$$\text{Set of Minima} = \cap_{k=0}^{\infty}\big(X \cap \{x \mid x'Qx + c'x \le \gamma_k\}\big)$$

with

$$\gamma_k \downarrow f^* = \inf_{x \in X} f(x).$$

Verify the condition $R_X \cap R \subset L$ of the preceding set intersection theorem, where R and L are the sets of common recession and lineality directions of the sets

$$\{x \mid x'Qx + c'x \le \gamma_k\}$$

Q.E.D.

CLOSURE UNDER LINEAR TRANSFORMATION

- Let C be a nonempty closed convex, and let A be a matrix with nullspace $N(A)$.

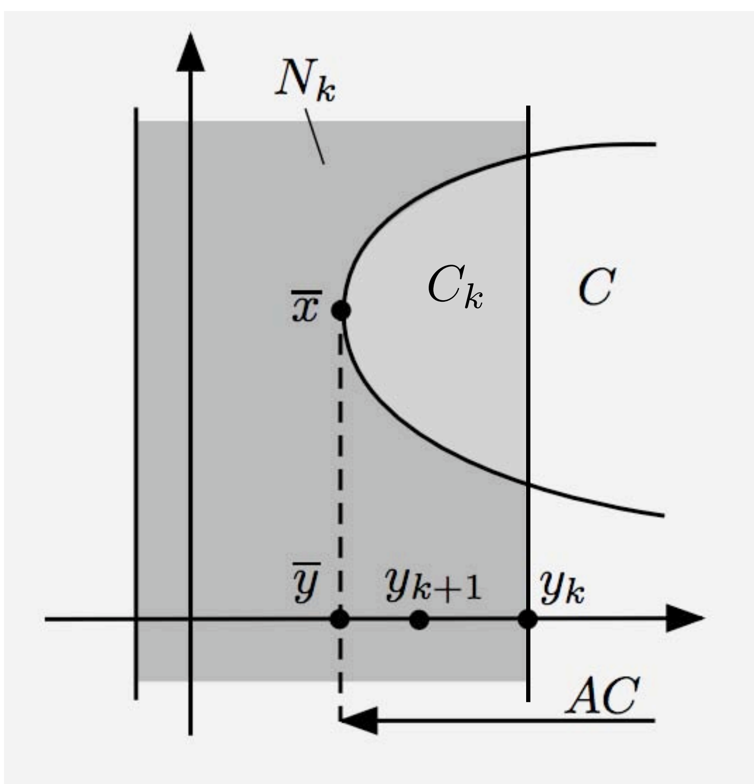
  (a) AC is closed if $R_C \cap N(A) \subset L_C$.

  (b) $A(X \cap C)$ is closed if X is a retractive set and

$$R_X \cap R_C \cap N(A) \subset L_C,$$

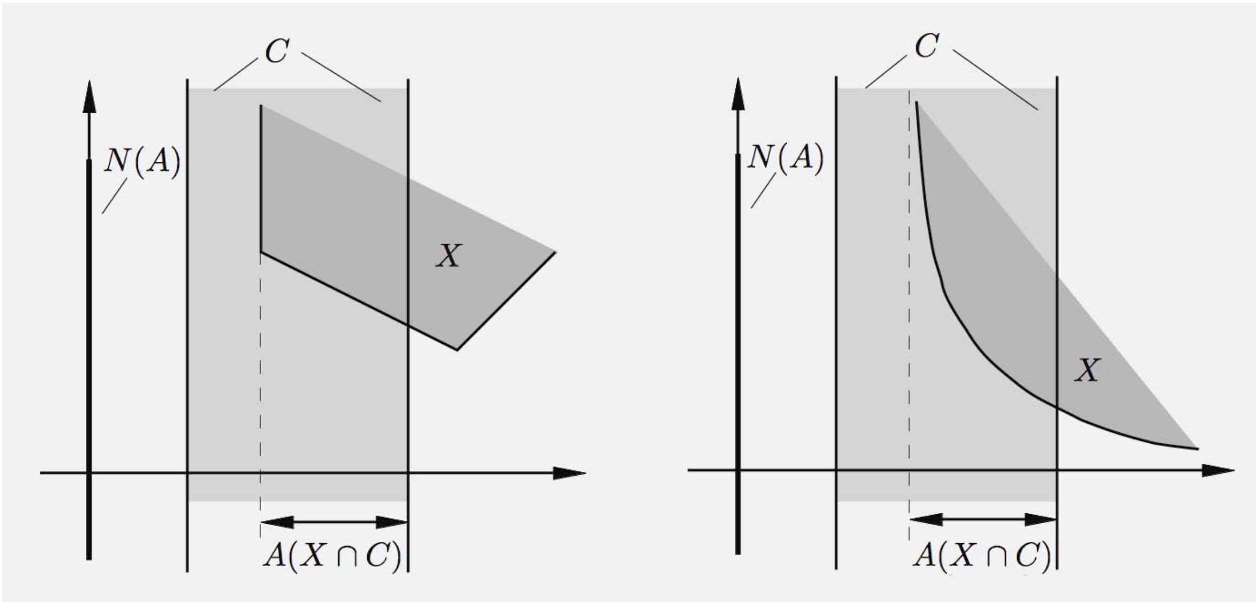
Proof: (Outline) Let $\{y_k\} \subset AC$ with $y_k \to y$. We prove $\cap_{k=0}^{\infty} C_k \neq \emptyset$, where $C_k = C \cap N_k$, and

$$N_k = \{x \mid Ax \in W_k\}, \quad W_k = \big\{z \mid \|z-y\| \leq \|y_k-y\|\big\}$$



- **Special Case:** AX is closed if X is polyhedral.

NEED TO ASSUME THAT X IS RETRACTIVE



Consider closedness of $A(X \cap C)$

- In both examples the condition

$$R_X \cap R_C \cap N(A) \subset L_C$$

is satisfied.

- However, in the example on the right, X is not retractive, and the set $A(X \cap C)$ is not closed.

CLOSEDNESS OF VECTOR SUMS

- Let $C_1, \ldots, C_m$ be nonempty closed convex subsets of $\Re^n$ such that the equality $d_1 + \cdots + d_m = 0$ for some vectors $d_i \in R_{C_i}$ implies that $d_i = 0$ for all $i = 1, \ldots, m$. Then $C_1 + \cdots + C_m$ is a closed set.

- **Special Case:** If C_1 and $-C_2$ are closed convex sets, then $C_1 - C_2$ is closed if $R_{C_1} \cap R_{C_2} = \{0\}$.

Proof: The Cartesian product $C = C_1 \times \cdots \times C_m$ is closed convex, and its recession cone is $R_C = R_{C_1} \times \cdots \times R_{C_m}$. Let A be defined by

$$A(x_1, \ldots, x_m) = x_1 + \cdots + x_m$$

Then

$$A\,C = C_1 + \cdots + C_m,$$

and

$$N(A) = \big\{(d_1, \ldots, d_m) \mid d_1 + \cdots + d_m = 0\big\}$$

$$R_C \cap N(A) = \Big\{(d_1, \ldots, d_m) \mid d_1 + \cdots + d_m = 0,\ d_i \in R_{C_i},\ \forall\, i\Big\}$$

By the given condition, $R_C \cap N(A) = \{0\}$, so $A\,C$ is closed. **Q.E.D.**