LECTURE 2

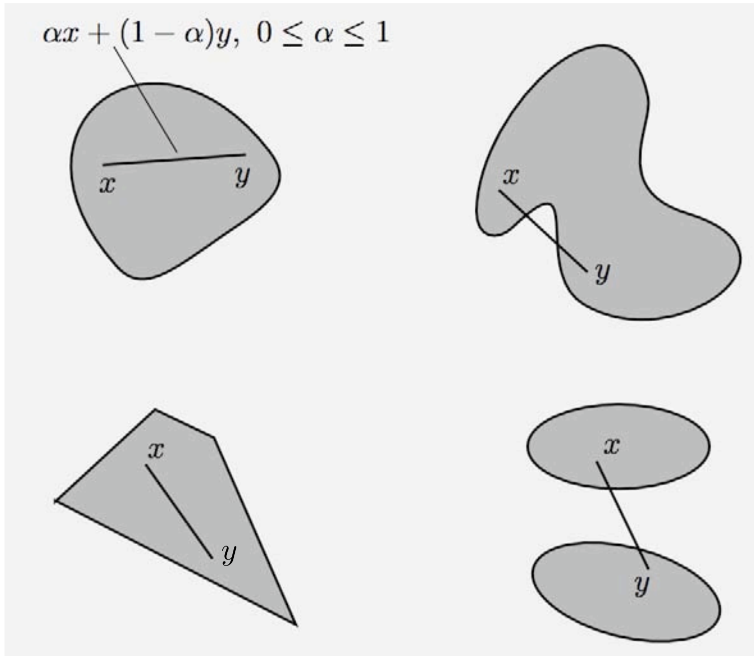
LECTURE OUTLINE

- Convex sets and functions
- Epigraphs
- Closed convex functions
- Recognizing convex functions

SOME MATH CONVENTIONS

- All of our work is done in $\Re^n$: space of n-tuples $x = (x_1, \ldots, x_n)$
- All vectors are assumed column vectors
- "$'$" denotes transpose, so we use x' to denote a row vector
- $x'y$ is the inner product $\sum_{i=1}^n x_i y_i$ of vectors x and y
- $\|x\| = \sqrt{x'x}$ is the (Euclidean) norm of x. We use this norm almost exclusively
- See the textbook for an overview of the linear algebra and real analysis background that we will use. Particularly the following:
  - Definition of sup and inf of a set of real numbers
  - Convergence of sequences (definitions of $\liminf$, $\limsup$ of a sequence of real numbers, and definition of lim of a sequence of vectors)
  - Open, closed, and compact sets and their properties
  - Definition and properties of differentiation

CONVEX SETS



- A subset C of $\Re^n$ is called *convex* if

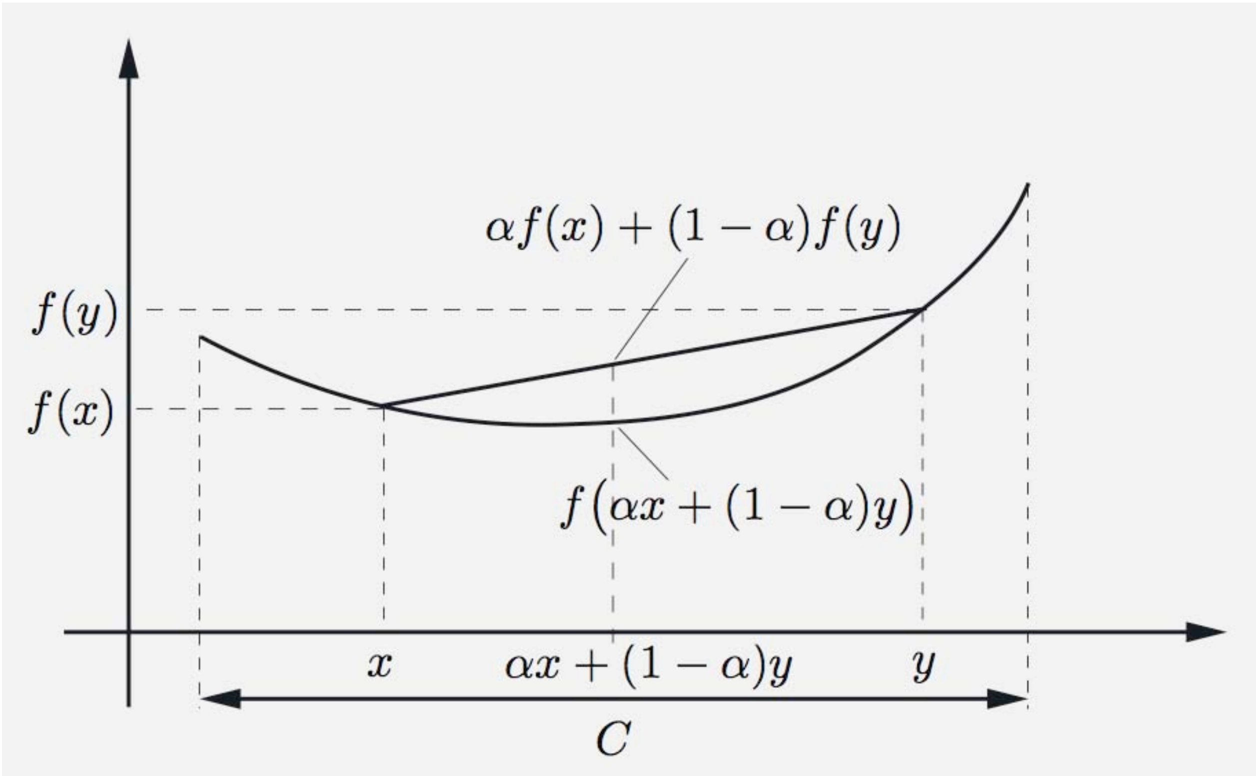
$$\alpha x + (1-\alpha)y \in C, \qquad \forall\, x, y \in C,\ \forall\, \alpha \in [0,1]$$

- Operations that preserve convexity
  - Intersection, scalar multiplication, vector sum, closure, interior, linear transformations
- Special convex sets:
  - **Polyhedral sets:** Nonempty sets of the form

$$\{x \mid a_j' x \le b_j,\ j = 1, \ldots, r\}$$

    (always convex, closed, not always bounded)
  - **Cones:** Sets C such that $\lambda x \in C$ for all $\lambda > 0$ and $x \in C$ (not always convex or closed)

CONVEX FUNCTIONS



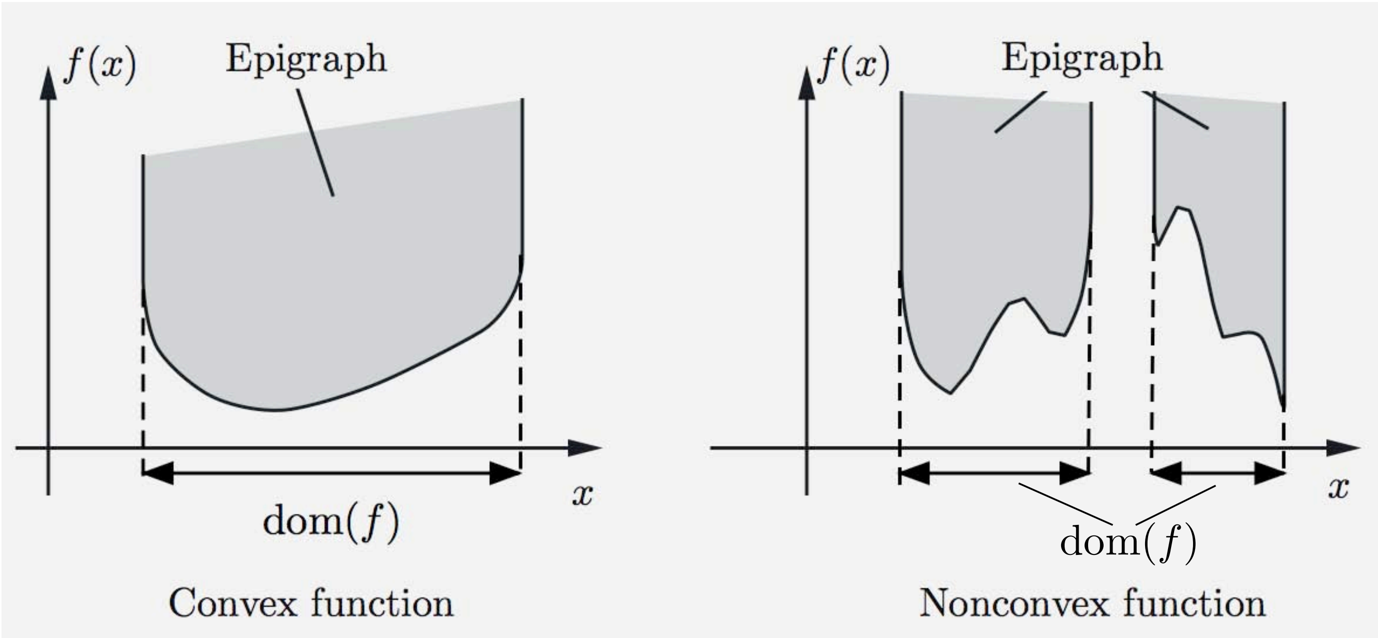
- Let C be a convex subset of $\Re^n$. A function $f : C \mapsto \Re$ is called *convex* if for all $\alpha \in [0,1]$

$$f\big(\alpha x+(1-\alpha)y\big) \leq \alpha f(x)+(1-\alpha)f(y), \quad \forall\, x, y \in C$$

If the inequality is strict whenever $a \in (0,1)$ and $x \neq y$, then f is called *strictly convex* over C.

- If f is a convex function, then all its level sets $\{x \in C \mid f(x) \leq \gamma\}$ and $\{x \in C \mid f(x) < \gamma\}$, where γ is a scalar, are convex.

EXTENDED REAL-VALUED FUNCTIONS



- The *epigraph* of a function $f : X \mapsto [-\infty, \infty]$ is the subset of $\Re^{n+1}$ given by

$$\text{epi}(f) = \big\{(x, w) \mid x \in X,\ w \in \Re,\ f(x) \le w\big\}$$

- The *effective domain* of f is the set

$$\text{dom}(f) = \big\{x \in X \mid f(x) < \infty\big\}$$

- We say that f is *convex* if epi(f) is a convex set. If $f(x) > -\infty$ for all $x \in X$ and X is convex, the definition "coincides" with the earlier one.

- We say that f is *closed* if epi(f) is a closed set.

- We say that f is *lower semicontinuous* at a vector $x \in X$ if $f(x) \le \liminf_{k\to\infty} f(x_k)$ for every sequence $\{x_k\} \subset X$ with $x_k \to x$.

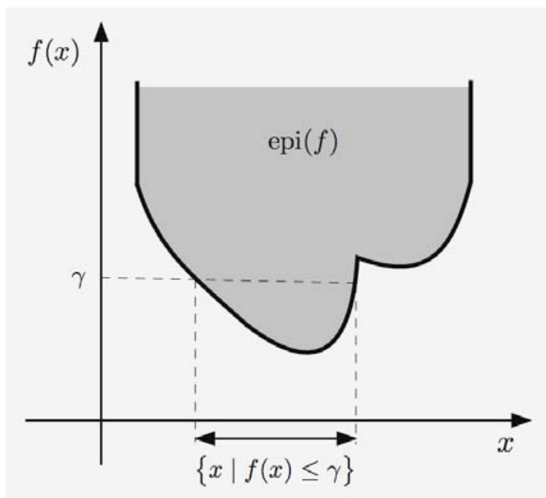
CLOSEDNESS AND SEMICONTINUITY I

- *Proposition*: For a function $f : \Re^n \mapsto [-\infty, \infty]$, the following are equivalent:

  (i) $V_\gamma = \{x \mid f(x) \leq \gamma\}$ is closed for all $\gamma \in \Re$.

  (ii) f is lower semicontinuous at all $x \in \Re^n$.

  (iii) f is closed.



- (ii) $\Rightarrow$ (iii): Let $\{(x_k, w_k)\} \subset \text{epi}(f)$ with $(x_k, w_k) \to (x, w)$. Then $f(x_k) \leq w_k$, and

$$f(x) \leq \liminf_{k \to \infty} f(x_k) \leq w \quad \text{so } (x, w) \in \text{epi}(f)$$

- (iii) $\Rightarrow$ (i): Let $\{x_k\} \subset V_\gamma$ and $x_k \to x$. Then $(x_k, \gamma) \in \text{epi}(f)$ and $(x_k, \gamma) \to (x, \gamma)$, so $(x, \gamma) \in \text{epi}(f)$, and $x \in V_\gamma$.

- (i) $\Rightarrow$ (ii): If $x_k \to x$ and $f(x) > \gamma > \liminf_{k \to \infty} f(x_k)$ consider subsequence $\{x_k\}_\mathcal{K} \to x$ with $f(x_k) \leq \gamma$ - contradicts closedness of V_γ.

CLOSEDNESS AND SEMICONTINUITY II

- Lower semicontinuity of a function is a "domain-specific" property, but closeness is not:
  - If we change the domain of the function without changing its epigraph, its lower semicontinuity properties may be affected.
  - **Example:** Define $f : (0,1) \to [-\infty, \infty]$ and $\hat{f} : [0,1] \to [-\infty, \infty]$ by

$$f(x) = 0, \qquad \forall\, x \in (0,1),$$

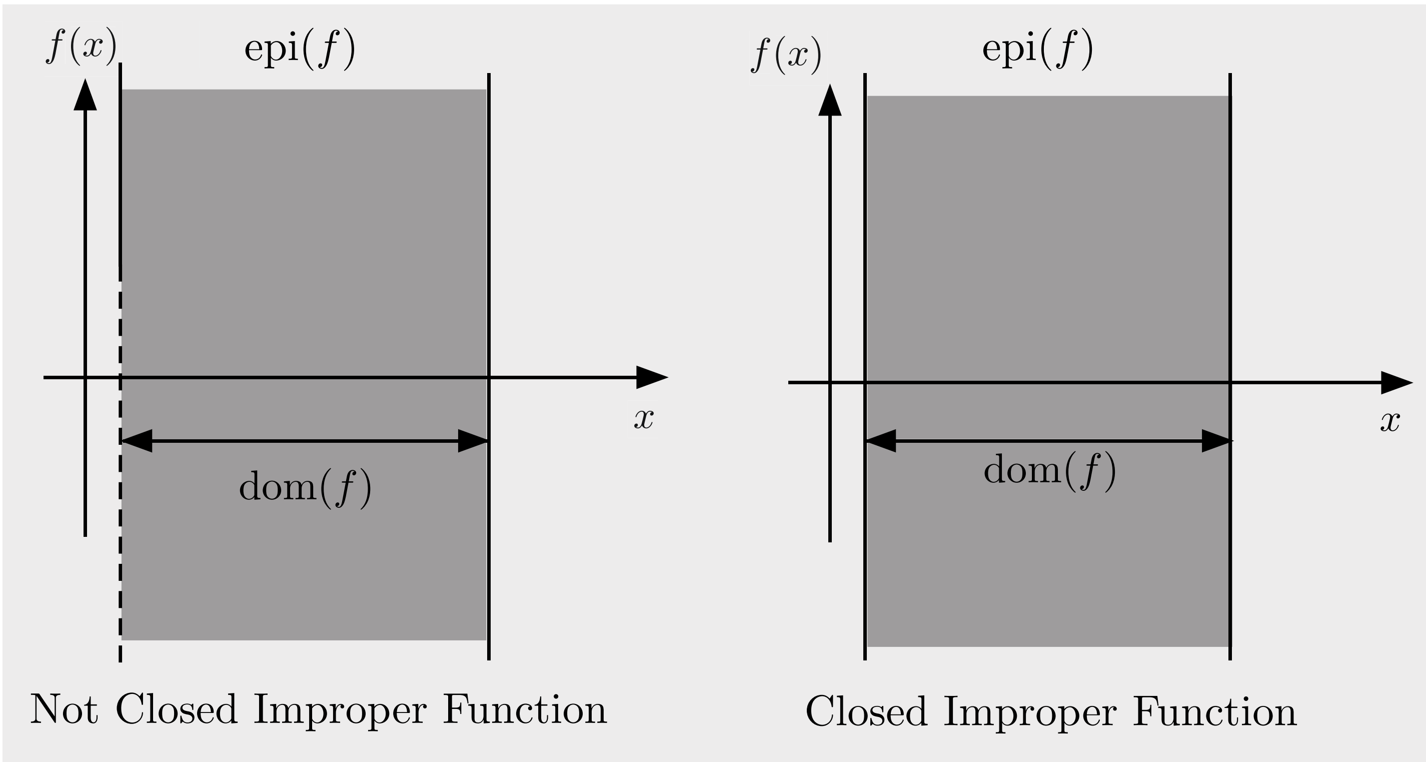
$$\hat{f}(x) = \begin{cases} 0 & \text{if } x \in (0,1), \\ \infty & \text{if } x = 0 \text{ or } x = 1. \end{cases}$$

  Then f and $\hat{f}$ have the same epigraph, and both are not closed. But f is lower-semicontinuous while $\hat{f}$ is not.

- Note that:
  - If f is lower semicontinuous at all $x \in \mathrm{dom}(f)$, it is not necessarily closed
  - If f is closed, $\mathrm{dom}(f)$ is not necessarily closed

- *Proposition*: Let $f : X \mapsto [-\infty, \infty]$ be a function. If $\mathrm{dom}(f)$ is closed and f is lower semicontinuous at all $x \in \mathrm{dom}(f)$, then f is closed.

PROPER AND IMPROPER CONVEX FUNCTION



- We say that f is *proper* if $f(x) < \infty$ for at least one $x \in X$ and $f(x) > -\infty$ for all $x \in X$, and we will call f *improper* if it is not proper.

- Note that f is proper if and only if its epigraph is nonempty and does not contain a "vertical line."

- An improper *closed* convex function is very peculiar: it takes an infinite value (∞ or $-\infty$) at every point.

RECOGNIZING CONVEX FUNCTIONS

- Some important classes of elementary convex functions: Affine functions, positive semidefinite quadratic functions, norm functions, etc.

- *Proposition*: Let $f_i : \Re^n \mapsto (-\infty, \infty]$, $i \in I$, be given functions (I is an arbitrary index set).

(a) The function $g : \Re^n \mapsto (-\infty, \infty]$ given by

$$g(x) = \lambda_1 f_1(x) + \cdots + \lambda_m f_m(x), \qquad \lambda_i > 0$$

is convex (or closed) if $f_1, \ldots, f_m$ are convex (respectively, closed).

(b) The function $g : \Re^n \mapsto (-\infty, \infty]$ given by

$$g(x) = f(Ax)$$

where A is an $m \times n$ matrix is convex (or closed) if f is convex (respectively, closed).

(c) The function $g : \Re^n \mapsto (-\infty, \infty]$ given by

$$g(x) = \sup_{i \in I} f_i(x)$$

is convex (or closed) if the f_i are convex (respectively, closed).

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