The Method of Moments in Electromagnetics

Massachusetts Institute of Technology
6.635 lecture notes

1 Introduction

In the previous lecture, we wrote the EFIE for an incident TE plane wave on a PEC surface. The solution was then obtained by some types of "intuitive" arguments, such as dividing the integration domain into small elements and supposing that the unknown does not vary too much over each elementary cell.

We shall now see more rigorously what we actually did, and show that it was in fact a simple version of the Method of Moments.

R. F. Harrington was the first to use the method of moments (MoM) in electromagnetics and his book remains a fundamental reference (and very easy to read!):

R. F. Harrington, "*Field Computation by Moment Method*" (is now available from IEEE Press).

2 PEC surface with TE incident wave: EFIE

The situation we studied last time is depicted in Fig. 1. The integral equation (EFIE) we

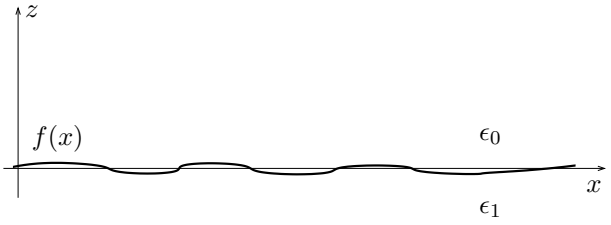

Figure 1: Rough surface S separating two media.

eventually obtained was:

$$\Phi_{\text{inc}}(\bar{r}) = \int_{-L/2}^{L/2} \mathrm{d}x' \sqrt{1 + \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2}\, g(x, f(x), x', f(x'))\, [\hat{n} \cdot \nabla\Phi(\bar{r}')]|_{z'=f(x')}\,, \tag{1}$$

which we rewrote as

$$b(x) = \int_{-L/2}^{L/2} \mathrm{d}x' K(x, x')\, U(x')\,, \tag{2}$$

where

$$\sqrt{1+\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2}\,[\hat{n}\cdot\nabla\Phi(\bar{r}')]|_{z'=f(x')} = U(x')\,, \tag{3a}$$

$$\Phi_{\mathrm{inc}}(x,f(x)) = b(x)\,, \tag{3b}$$

$$K(x,x') = g(x,f(x),x',f(x'))\,, \tag{3c}$$

where $U(x')$ is the unknown we are solving for.

The important step we did from this integral, although it probably appeared straightforward, was to say that x can only take discrete values on the surface, thus defining N intervals of length Δx:

$$x \in \{x_i\}\,, \quad i = 1,\ldots,N, \tag{4}$$

The assumptions were therefore:

1. $x \in \{x_i\}\,, \quad i = 1,\ldots,N.$
2. $U(x')$ is constant on each interval.

which eventually yielded the following system of equations (supposing that the problem related to the singularity of the Green's function has been accounted for):

$$\sum_{n=1}^{N} A_{mn}\,U_n = b_m\,. \tag{5}$$

This is a matrix equation with the two indices m and n corresponding to:

m: observation point $\rightarrow$ unprimed coordinates.

n: source point $\rightarrow$ primed coordinates.

Physical interpretation:

- element (m,n) represents the effect of cell n on cell m.
- element (m,m) represents the self-term.

Mathematically:

The steps we had to perform to from Eq. (2) to Eq. (5) are

1. Write the unknown as
$$U(x') = \sum_n U_n \delta(x_n)\Delta x\,, \tag{6}$$
stating that now the unknowns become $\{U_n\}$ which are the amplitudes of the function. The integral equation becomes:
$$\int_{-L/2}^{L/2} \mathrm{d}x' K(x,x')U(x') = \Delta x \sum_n U_n K(x,x_n) = b(x)\,. \tag{7}$$

2. Dot-multiply both sides by $\delta(x_m)$:

$$\Delta x \sum_n U_n \, K(x_m, x_n) \; = \; b(x_m). \tag{8}$$

These two steps are at the basis of the method of moments:

1. First step can actually be decomposed into two steps:
   (a) Mesh the structure (*i.e.* choose the intervals over which U_n will be defined).
   (b) Expand the unknown $U(x)$ into BASIS FUNCTIONS.
2. The second steps concerns the observation: dot-multiply both sides of the equation by a TEST FUNCTION (or WEIGHTING FUNCTION).

In the previous example:

- basis functions: pulse basis function.
- testing functions: we point-patch the integral equation at $x = x_m$ (the method is therefore called *point matching*).

This is a very simple, yet very widely used version of the method of moments.

3 General considerations on MoM

Let us consider the inhomogeneous equation:

$$\mathcal{L}(f) = g \, , \tag{9}$$

where $\mathcal{L}$ is a linear operator, g is known, and f is to be determined. We shall now perform the two essential steps we have highlighted above.

1. Let f be expanded in a series of functions:

$$f = \sum_n \alpha_n \, f_n \, , \tag{10}$$

   where α_n are constant. The set f_n is called *expansion function*, or *basis functions*.

   Note that for an exact solution, the summation should be taken to ∞, but has to be truncated in practice.

2. It is assumed that a suitable inner product has been defined for the problem. Now, we define a set of *weighting functions*, or *testing functions*, $w_1 \, , \; w_2 \, , \; \ldots \, , w_N$ in the range of $\mathcal{L}$, and take the inner product of the previous equation with w_m:

$$\sum_n \alpha_n < w_m, \mathcal{L} f_n > = < w_m, g > \; . \tag{11}$$

The system can now be written in matrix form as:

$$[A_{mn}]\,[\alpha_n] \;=\; [g_m]\,, \tag{12}$$

where

$$[A_{mn}] = \begin{pmatrix} < w_1, \mathcal{L}f_1 > & < w_1, \mathcal{L}f_2 > & \dots \\ < w_2, \mathcal{L}f_1 > & < w_2, \mathcal{L}f_2 > & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad [\alpha_n] = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \end{pmatrix}, \quad [g_m] = \begin{pmatrix} < w_1, g > \\ < w_2, g > \\ \vdots \end{pmatrix}. \tag{13}$$

If the matrix $[A_{mn}]$ is not singular, the unknowns α_n are simply given by:

$$[\alpha_n] = [A_{mn}]^{-1}\,[g_m]\,, \tag{14}$$

and the original function f can be reconstructed using Eq. (10). We can now generalize the following definitions:

- The basis functions used previously are defined as:

$$\text{Pulse basis functions: } f_n = \begin{cases} 1 & \text{if } x \text{ belongs to the interval } n \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

- The testing (or weighting functions):

Point matching = taking Dirac δ functions as testing functions.

4 A simple example for electrostatic

The example is taken from the reference mentioned at the beginning of this document.

Let us consider a square plate of side $2a$ lying on the $z = 0$ plane with its center at the origin (see Fig. 2. Let $\sigma(x, y)$ represent the surface density on the plate, assumed to have zero thickness. The electrostatic potential Φ at any point in space is given by

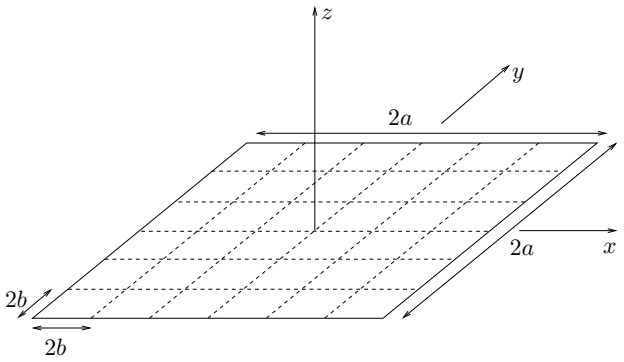

Figure 2: Discretized square plate.

$$\Phi(x,y,z) = \int_{-a}^{a} \mathrm{d}x' \int_{-a}^{a} \mathrm{d}y'\ \sigma(x',y')\, g(\bar{r},\bar{r}')\,, \tag{16a}$$

$$g(\bar{r},\bar{r}') = \frac{1}{4\pi\epsilon R}\,, \tag{16b}$$

$$R = \sqrt{(x-x')^2 + (y-y')^2 + z^2}\,. \tag{16c}$$

The boundary condition is $\Phi = V =$ constant on the plate. The integral equation for the problem is therefore

$$V = \int_{-a}^{a} \mathrm{d}x' \int_{-a}^{a} \mathrm{d}y' \frac{1}{4\pi\epsilon}\ \frac{\sigma(x',y')}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}}\,, \tag{17}$$

where the unknown to determine is $\sigma(x',y')$. Let us perform the three steps mentioned before:

1. Mesh the structure: divide the plate into N squares of size $2b$ (see Fig. 2).

2. Basis functions: let us choose

$$\sigma(x',y') \simeq \sum_{n=1}^{N} \alpha_n\, f_n(x',y') \qquad \text{with } f_n(x',y') = \begin{cases} 1 & \text{on } \Delta S_n \\ 0 & \text{on } \Delta S_m,\ m \neq n. \end{cases} \tag{18}$$

3. Test functions: we choose to satisfy the integral equation at the mid-point (x_m, y_m) of each ΔS_m:

$$w_m = \delta(x - x_m)\delta(y - y_m). \tag{19}$$

With these three steps, we construct the matrix as ($z = 0$):

$$[A_{mn}] = \int_{\Delta x_n} \mathrm{d}x' \int_{\Delta y_n} \mathrm{d}y'\ \frac{1}{4\pi\epsilon}\ \frac{1}{\sqrt{(x_m - x')^2 + (y_m - y')^2}}\,. \tag{20}$$

It is obvious to see that this integral is singular at $(x_m, y_m) \in \Delta S_m$. In this simple case fortunately, we can perform the integration analytically (this is not always the case), and write:

$$A_{nn} = \int_{-b}^{b} \mathrm{d}x' \int_{-b}^{b} \mathrm{d}y' \frac{1}{4\pi\epsilon}\ \frac{1}{\sqrt{x'^2 + y'^2}}\ = \frac{2b}{\pi\epsilon}\ \ln(1+\sqrt{2})\,, \tag{21a}$$

$$A_{mn} \simeq \frac{\Delta S_n}{4\pi\epsilon R_{mn}} = \frac{b^2}{\pi\epsilon\sqrt{(x_m - x')^2 + (y_m - y')^2}}\,, \qquad m \neq n\,. \tag{21b}$$

To rewrite this with the language of linear space:

$$f(x,y) = \sigma(x,y)\,, \tag{22a}$$

$$g(x,y) = V \text{ on the plate (the discretization gives: } g_m = \begin{pmatrix} V \\ V \\ \vdots \end{pmatrix}) \tag{22b}$$

$$\mathcal{L}(f) = \int_{-a}^{a} \mathrm{d}x' \int_{-a}^{a} \mathrm{d}y' \frac{1}{4\pi\epsilon} \frac{f(x',y')}{\sqrt{(x-x')^2 + (y-y')^2}}\,. \tag{22c}$$

Note: if we add another plate under the existing one, at $z = -2d$, with a potential $-V$, we build a new problem that can be analyzed in two ways:

1. By meshing both plates (*i.e.* meshing everything, which is of course always possible). This will yield matrix twice as big as the previous one, to solve for twice as many unknowns.

2. By using the image theory, and saying that the new problem is equivalent to the one of a unique plate on top of a ground plane at $z = -d$. In that case, we only have to change the Green's function to take the ground plane into account, and we keep the same number of unknowns as in the initial problem. When possible, this solution is better because computationally less expensive (analytically more expensive).

Basically, a general trend is to have a Green's function that represents as much as possible the environment and to mesh only those parts that are external to the environment. This is in fact the reason why people are looking for Green's functions in layered media, periodic media, etc.

5 Vectorial MoM

We can of course apply the MoM to the vectorial case, like for example the equation:

$$\bar{E}(\bar{r}) = i\omega\mu \int_S \mathrm{d}s' \, \overline{\overline{G}}_e(\bar{r}, \bar{r}') \cdot \bar{J}(\bar{r}') \,. \tag{23}$$

The general expansion of the current will be:

$$\bar{J}(\bar{r}') = \sum_n \alpha_n \bar{f}_n(\bar{r}') \,, \tag{24}$$

yielding

$$\bar{E}(\bar{r}) = i\omega\mu \sum_n \alpha_n \int_{S_n} \mathrm{d}s' \, \overline{\overline{G}}_e(\bar{r}, \bar{r}') \cdot \bar{f}_n(\bar{r}') \,. \tag{25}$$

The third step is to dot-multiply the equation with a testing function $\bar{h}_m$ and integrate over the cell surface (*i.e.* perform the inner product):

$$\int_{S_m} \bar{E}(\bar{r}) \cdot \bar{h}_m(\bar{r}) = i\omega\mu \sum_n \alpha_n \int_{S_n} \mathrm{d}s' \int_{S_m} \mathrm{d}s \, \bar{h}_m(\bar{r}) \cdot \overline{\overline{G}}_e(\bar{r}, \bar{r}') \cdot \bar{f}_n(\bar{r}') \,. \tag{26}$$

The double double-integral on the right-hand side of this equation is known as the "impedance term" since we can cast this system of equation into a matrix representation as:

$$[E_m] = [Z_{mn}] \, [\alpha_n] \,. \tag{27}$$

6 Other basis and testing functions

The advantage of the MoM over purely numerical methods is that there is still a large part that remains analytic (like the Green's functions for example). Yet, it remains a numerical method based on a matrix inversion technique and therefore, convergence issues need to be examined.

The convergence of the MoM is closely related to the choice of basis functions and, although to a lesser extend, to the choice of testing functions. There are essentially two families of basis functions:

1. Entire domain basis functions: using these functions to expand the unknowns is analogous to a Fourier expansion or to a modal expansion. These types of functions yield a good convergence of the method but are not versatile since the geometry need be regular in order to have the modes defined. Note that in this case there is no use to mesh the geometry.

2. Sub-domain basis functions: they rely on a proper meshing of the geometry, which can be rectangular, triangular, etc. The choice of basis functions is here very wide, from Dirac δ (like for the weighting functions shown in this document), pulses (basis functions shown in this document), piecewise linear, etc.

Finally, we can mention that point matching, which is easy to grasp and straightforward to implement, may not yield an optimal convergence. In most of the applications, the Galerkin technique is better, which consists in choosing the same testing functions as the basis functions. This applies to both sub-domain and entire domain functions.