Application Example 13

(Propagation of uncertainty through linear formulas: second-moment analysis)

DESIGN LOAD FACTORS FOR STRUCTURAL COLUMNS

Load factors are numbers greater than 1 by which nominal loads are multiplied to provide sufficient safety in structural design. Design codes allow engineers to use different design load factors on structural columns depending on the number of floors supported by the column. The factors typically decrease as the number of supported floors increases. This might sound counter-intuitive, since one would expect the code to require a higher level of safety for taller buildings. The uncertainty analysis that follows explains the rationale for such code provision.
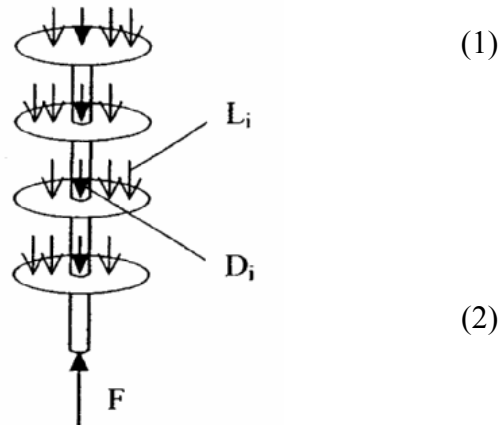
Second Moment Analysis of Column Load

Suppose that a structural column supports n floors and that the load on the column, F, comes primarily from dead loads (D) and live occupancy loads (L). Therefore, the total load on the column is

$$F = D + L \tag{1}$$

where

$$D = \sum_{i=1}^{n} D_i \tag{2}$$

$$L = \sum_{i=1}^{n} L_i$$

and D_i and L_i are the dead and live loads contributed by floor i. In most cases, dead loads are much more accurately known that live loads are. Therefore, D may be considered

deterministically known. A second simplifying assumption we make is that the live loads L_i have the same marginal distribution (hence they have the same mean value m, the same variance σ^2, and the same coefficient of variation $V = \sigma/m$ for all floors). Finally, we assume that the loads L_i are "equicorrelated", meaning that the correlation coefficient between L_i and L_j is the same for all $i \neq j$. We denote such correlation coefficient by ρ. Under these conditions, the mean value, variance and coefficient of variation of L in Eq. 2 are

$$
\begin{aligned}
m_L &= nm \\
\sigma_L^2 &= n\sigma^2 + n(n-1)\rho\sigma^2 = n^2\sigma^2\left[\frac{1-\rho}{n} + \rho\right] \\
V_L &= \frac{\sigma_L}{m_L} = V\sqrt{\frac{1-\rho}{n} + \rho}
\end{aligned}
\tag{3}
$$

Design Load F*

The question we want to address is, what is a reasonable load value F* to be used for design? Under the current assumptions, the mean value and variance of F are

$$
\begin{aligned}
m_F &= D + m_L = D + nm \\
\sigma_F^2 &= \sigma_L^2 = n^2\sigma^2\left[\frac{1-\rho}{n} + \rho\right]
\end{aligned}
\tag{4}
$$

and a sensible way to express F* is $F^* = m_F + \gamma\sigma_F$ where γ is a factor of safety with typical values between 3 and 5, depending on the importance of the building. Using Eq. 4, F* may be expressed as

$$
\begin{aligned}
F^* &= D + m_L + \gamma m_L V_L \\
&= D + (1 + \gamma V_L)m_L
\end{aligned}
\tag{5}
$$

Equivalently, one may set $\beta = (1 + \gamma V_L)$ and write Eq. 5 as

$$F^* = D + \beta m_L \tag{6}$$

Notice that, while the factor γ is applied to the standard deviation of L, the factor β is applied to the mean value of L.

Factor β in Eq. 6

It is interesting to see how β in Eq. 6 varies with the number of supported floors n, the correlation coefficient ρ between the live loads on different floors and the coefficient of variation V of the live loads L_i, in addition to the factor γ. Using Eq. 3,

$$\begin{aligned} \beta &= 1 + \gamma V_L \\ &= 1 + \gamma V \sqrt{\frac{1-\rho}{n} + \rho} \end{aligned} \tag{7}$$

While the dependence of β on γ and V is evident, that on n and ρ is more complicated. Noticeable features of Eq. 7 are:

(i) For $\rho = 0$ (uncorrelated live loads), $\beta = 1 + \gamma V/\sqrt{n}$. This shows that, if one aimed at achieving the same reliability irrespective of the number of floors, then one would have to take $\beta - 1 \propto \frac{1}{\sqrt{n}}$, with $\beta \to 1$ as $n \to$. This limiting case makes sense, because for $\rho = 0$ and $n \to \infty$, Eq. 3 gives $V_L \to 0$.

(ii) For $\rho > 0$ (positively correlated live loads, which is a typical case in practice), the term $\sqrt{\frac{1-\rho}{n} + \rho}$ in Eq. 7 approaches the positive value $\sqrt{\rho}$ as $n \to \infty$. The reason why the asymptotic value of β is greater than 1 is that, when $\rho > 0$, V_L does not vanish asymptotically as $n \to \infty$.

(iii) For $\rho = 1$ (identical live loads on different floors), V_L does not depend on n. Then also β is independent of the number of floors n, with value $\beta = 1 + \gamma V$;

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(iv) $\rho < 0$. Although in the present problem negative correlations are unlikely to occur, it is nevertheless interesting to see what happens mathematically in this case. Notice first that, for $\rho < -1/(n-1)$, the factor $\left[\dfrac{1-\rho}{n}+\rho\right]$ in Eq. 3 becomes negative, producing a negative value of σ_L^2. This is of course impossible, because the variance cannot be negative. Indeed, in the case of n equicorrelated variables, ρ must be larger than $-1/(n-1)$. As $n \to \infty$, this condition becomes $\rho > 0$.

Problem 13.1

(a) Plot the factor β in Eq. 7 against n (say, for n from 1 to 30) for $\gamma = 3$, $V_L = 0.1$ and three values of the correlation coefficient, $\rho = 0, 0.1, 0.3$ (total of 3 plots, one for each ρ). Comment on the results.

(b) For certain buildings and certain types of occupancy, the assumption of equicorrelated live loads on different floors may not be appropriate. Consider as an alternative a live-load model with floor loads L_i and L_j that have an exponentially decaying correlation, of the type

$$Corr[L_i, L_j] = \rho^{|i-j|} \tag{8}$$

where ρ is a positive number less than 1. According to this model, live loads on widely separated floors are less correlated than live loads on adjacent floors. Modify the analysis for this case. Produce plots of β against n for $\gamma = 3$, $V_L = 0.1$ and $\rho = 0$, 0.1, 0.3. Compare with the equicorrelated case.

Notice that the variation of β with n is ultimately related to the fact that the correlation coefficient of the sum of non-perfectly dependent random variables decreases as the number of added variables increases. This result is also at the basis of other important observations, for example that the variability in monthly precipitation, when measured

through the coefficient of variation, is much smaller than the variability in daily precipitation. Another example of the reduction in variability when summing or averaging many variables is the reduction in the coefficient of variation of the strength of large material specimens relative to smaller specimens.

Problem 13.2

Formulate a problem from your area of professional interest or from everyday life, in which the variability of the sum of several variables matters. Judgmentally assign mean values, variances, and correlation coefficients to the variables that are being added. Calculate the mean value, variance, and coefficient of variation of the quantity of interest (the sum). Comment on the results.