

## Quiz #1 Solutions

### Problem 1

Define the following events:

D: dangerous conditions

$D^c$ : normal conditions

T: alarms indicated dangerous conditions

$T^c$ : alarm indicates normal conditions

Given:

$$P[T | D] = 0.95 \quad P[T^c | D] = 0.05$$

$$P[T | D^c] = 0.005 \quad P[T^c | D^c] = 0.995$$

$$P[D] = 0.005 \quad P[D^c] = 0.995$$

(a) Probability of false alarm:

$$P[D^c | T] = P[D^c] \cdot \frac{P[T | D^c]}{P[T]}$$

$P[T]$  is found using the Total Probability Theorem as

$$P[T] = P[T | D] \cdot P[D] + P[T | D^c] \cdot P[D^c]$$

Therefore,

$$P[D^c | T] = P[D^c] \cdot \frac{P[T | D^c]}{P[T]} = 0.5116$$

(b) Probability of unidentified critical condition:

$$P[D | T^c] = P[D] \cdot \frac{P[T^c | D]}{P[T^c]} = 0.0002525$$

(c) Number of False Alarms =  $P[D^c | T] \times P[T] \times 365 \times 10 = 18$  times

Number of unidentified dangerous conditions =  $P[D | T^C] \times P[T^C] \times 365 \times 10 = 1$  time

The alarm system therefore gives 18 false alarms in the 10 year period, which can be a hassle. It does however fail to identify only one dangerous condition, so performs well here. Good alarms aim to minimize false alarms and eliminate unidentified conditions.

## Problem 2

(a) The answer to this question is somewhat subjective, and so will be different depending on who answers, but here are a few ideas that should be included.

The three conditions that a point process need to satisfy to be Poisson are:

1. Stationarity. The probability that an event in a short interval  $(t, t + \Delta t)$  is approximately  $\lambda(\Delta t)$  for any  $t$ .
2. Non-Multiplicity. The probability of two or more events in  $(t, t + \Delta t)$  is negligible.
3. Independence (No Memory). The number of events in an interval of time is independent of the number in any other interval of time.

If one wants to model flood-producing storms as a Poisson process, they need to satisfy these three conditions. If one assumes that flood-producing storms are rare, and occur randomly in time, then one can argue that:

The condition of stationarity is satisfied, since the longer the time period one considers, the more likely a storm will occur.

The condition of non-multiplicity is satisfied because storms don't occur at the same time.

The most difficult condition to satisfy is that of independence. This assumes no memory, but weather patterns typically exhibit some persistence, i.e. seasons, and therefore do show dependence.

Assuming Poisson storm arrivals with  $\lambda = 1/20$  years.

Define:

Y: the number of storms in time,  $t$ . Y is a Poisson variable with PMF:

$$P_Y(y) = \frac{(\lambda t)^y e^{-\lambda t}}{y!}$$

The probability that the water height exceeds 3 m is given by:

$$P[h > 3] = 1 - F_H(3) = e^{-\frac{3}{2}}$$

The original Poisson Process is thinned with probability  $P[h > 3]$  to result in a new thinned Poisson Process for storms that cause the water height to exceed 3 m. The new rate of the process is  $\lambda^* = \lambda P[h > 3]$ .

Define:

$Y^*$ : the number of storms that cause water heights to exceed 3 m in time,  $t$ .  $Y^*$  is a Poisson variable with PMF:

$$P_{Y^*}(y^*) = \frac{(\lambda^* t)^{y^*} e^{-\lambda^* t}}{y^*!}$$

The probability that the water height will exceed 3 m at least once during the next 100 years is given by:

$$P[\text{At least one storm occurrence}] = 1 - P[\text{No storms}] = 1 - P_{Y^*}(0) = 1 - e^{-\lambda^* t} = 0.672$$

### Problem 3

Given the joint density  $f_{X_1, X_2}(x_1, x_2)$  is uniform inside the disk.

(a) The value of  $f_{X_1, X_2}(x_1, x_2)$  is obtained from knowing that the total volume under the  $f_{X_1, X_2}(x_1, x_2)$  cylinder must be equal to 1.

Therefore,

$$\text{Volume} = f_{X_1, X_2}(x_1, x_2) \times \pi r^2 = 1, \text{ and}$$

$$f_{X_1, X_2}(x_1, x_2) = \frac{1}{\pi} \text{ inside the disk,}$$

$$f_{X_1, X_2}(x_1, x_2) = 0 \text{ elsewhere.}$$

$X_1$  and  $X_2$  are not independent.

For  $X_1$  and  $X_2$  to be independent,  $f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$

Since,  $f_{X_1, X_2}(x_1, x_2) = \frac{1}{\pi}$  cannot be expressed as the product of two functions. So,  $X_1$  and  $X_2$  are dependent.

(b) The marginal PDF of  $X$ ,  $f_{X_1}(x_1)$  is obtained by integrating the joint density over all  $X_2$ .

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1, X_2}(x_1, x_2) dx_2$$

The limits of integration are given by the value of  $X_2$  at any  $X_1$ :

$$x_2 = \pm\sqrt{1-x_1^2}, \text{ and so:}$$

$$f_{X_1}(x_1) = \int_{-\sqrt{1-x_1^2}}^{\sqrt{1-x_1^2}} f_{X_1, X_2}(x_1, x_2) dx_2 = \frac{2}{\pi} \sqrt{1-x_1^2} \quad \text{for } -1 \leq x_1 \leq 1$$

$$f_{X_1}(x_1) = 0 \text{ otherwise}$$

#### Problem 4

By Linearity of expectation:

$$E[Y_1] = E[X_1] + E[X_2] = 2m$$

$$E[Y_2] = 2.E[X_1] = 2m$$

Using second moment analysis

$$\text{Var}[Y_1] = \text{Var}[X_1] + \text{Var}[X_2] + 2\text{Cov}[X_1, X_2] = 2\sigma^2 \text{ because of independence.}$$

$$\text{Var}[Y_2] = 2^2 \text{Var}[X_1] = 4\sigma^2$$

The vector  $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$  is a linear function of the vector  $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ , specifically:

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

If we let  $\underline{\mathbf{B}} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ , then using the results of second moment analysis:

$$\underline{\Sigma}_Y = \underline{\mathbf{B}} \underline{\Sigma}_X \underline{\mathbf{B}}^T = \begin{bmatrix} 2\sigma^2 & 2\sigma^2 \\ 2\sigma^2 & 4\sigma^2 \end{bmatrix}, \text{ and so } \text{Cov}[Y_1, Y_2] = 2\sigma^2.$$