1.033/1.57 Q#2: Elasticity Bounds – Conical Indentation Test

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Instrumented nano-indentation is a new technique in materials science and engineering to determine material properties at very fine scales. A typical indentation test is composed of a loading and an unloading part. The loading part is used to extract strength properties, the unloading portion is used to extract the elasticity properties of the indented material. We have already studied the link between strength properties and hardness measurements for a 'flat' indenter (Homework Set #2) and for a conical indenter (Quiz #1). This exercise deals with the link between elastic properties and the indentation result upon unloading.

An indentation test is a surface test, but its effect is felt in a bulk volume of characteristic size L around the indentation cone. Within this bulk zone the material undergoes deformation as a consequence of the indentiation, while the material situated outside this zone will 'not feel' the localized indentation (no deformation). The focus of this exercise is to estimate this characteristic size L associated with the elastic unloading by means of the upper bound displacement approach. To this end, we consider a rigid conical indenter of half-apex angle α, which –during the loading phase– has penetrated into the material to an indentation depth h (see figure 1 TOP). At this stage, we consider an infinitesimal unloading $|s| << h$. The slope of the unloading, $dF/ds > 0$ (see figure 1. BOTTOM), is related to the stiffness properties of the indented material by:

$$\frac{dF}{ds} = \frac{2}{\sqrt{\pi}}\sqrt{A}M \tag{1}$$

where $A = \pi R^2$ is the projected contact area at the surface $z = 0$ (see figure), and M is the indentation modulus. In this exercise, we will first establish an upper bound for the relation between M and the elasticity constants of a homogeneous, linear elastic isotropic material, characterized by the Lamé constants $\lambda, G\,(= const)$. Then, we will use this solution to evaluate the characteristic length scale L. Throughout this exercise we will assume quasi-static conditions (inertia effects neglected), and we will neglect body forces.

NOTE: PLEASE HAND IN THE FIGURE BELOW WITH THE REQUESTED DRAWINGS TOGETHER WITH YOUR SOLUTION. THANKS & GOOD LUCK.

1. **Kinematically Admissible Displacement Field**: For purpose of analysis, we consider two subdomains of finite size, noted respectively $\Omega_1 = \pi R^2 L$ and $\Omega_2 = \pi R^2 L \left[(1 + L/R)^2 - 1 \right.$ (see figure 1). Both domains are assumed to extend by a length L in both the vertical and the horizontal direction. The displacement is assumed to be non-zero only within Ω_1 and Ω_2. The displacement in the rest of the halfspace, i.e. situated outside $\Omega = \Omega_1 \cup \Omega_2$, is assumed to be zero. In Ω_1 and Ω_2, we consider vertical displacement fields of the form:

$$\text{in } \Omega_i : \boldsymbol{\xi}_i' = f_i(z, r)\, \mathbf{e}_z;\ i = 1, 2 \tag{2}$$

   (a) For both domains, Ω_1 and Ω_2, and their common interface, specify the conditions which functions $f_i(z, r)$ $(i = 1, 2)$ need to satisfy, so that $\boldsymbol{\xi}'$ is kinematically admissible. HINT: The indenter is a rigid cone, and the test is displacement driven.

   (b) We consider a linear form of the displacement in Ω_1 and Ω_2, that is:

$$\boldsymbol{\xi}_1' = f_1(z, r)\, \mathbf{e}_z = s \left(1 - \frac{z - h(1 - r/R)}{L} \right) \mathbf{e}_z \tag{3}$$

$$\boldsymbol{\xi}_2' = f_2(z, r)\, \mathbf{e}_z = s (1 - z/L) \left(1 - \frac{r - R}{L} \right) \mathbf{e}_z \tag{4}$$

   Sketch this displacement field in Figure 1, and show that it is kinematically admissible in $\Omega = \Omega_1 \cup \Omega_2$.

   (c) Determine the non-zero strain components, in Ω_1 and Ω_2. HINT: Go back to the geometrical interpretation of the linear strain components. By means of very precise mechanics argument, show that the displacement field $\boldsymbol{\xi}'$ is NOT the linear elastic solution $(\boldsymbol{\xi}, \boldsymbol{\sigma})$ of the problem.

2. **Potential Energy**: We will employ the upper bound energy method, $\mathcal{E}_{pot}(\boldsymbol{\xi}) \leq \mathcal{E}_{pot}(\boldsymbol{\xi}')$ [notation as in lecture notes]:

   (a) For the displacement driven indentation test, show that the potential energy of the solution of the problem reads $\mathcal{E}_{pot}(\boldsymbol{\xi}) = \frac{1}{2} \frac{dF}{ds} s^2$. Display this result in the force-displacement curve in Figure 1 (BOTTOM).

   (b) Using the displacement field $\boldsymbol{\xi}_i'$ defined by (3) and (4), show that the upper bound of the potential energy $\mathcal{E}_{pot}(\boldsymbol{\xi}')$ reads:

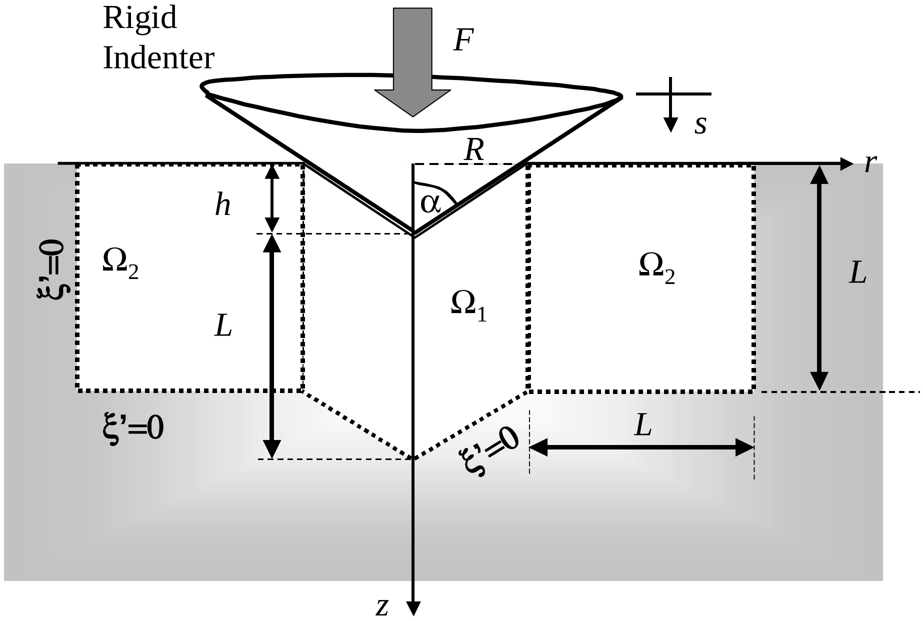
$$\mathcal{E}_{pot}(\boldsymbol{\xi}') = \pi L R^2 \left(\frac{s}{L} \right)^2 \frac{1}{12} \left[2(5\lambda + 12G) + (\lambda + 4G)(L/R) + 6G(h/R)^2 \right] \tag{5}$$

   (Hint: If you run out of time, just continue working in what follows with the result (5).

   (c) With the results of 2a. and 2b. develop an upper bound M^+ for the indentation modulus:

$$M \leq M^+(\lambda, G, \alpha, R/L) \tag{6}$$

   (d) Finally, the exact solution for the indentation stiffness is $M = E/(1 - \nu^2)$. From the previous results, determine an upper bound for the characteristic length scale L/R of the indentation test, and determine L/R for $\alpha = \pi/4$ and $\nu = 0$.



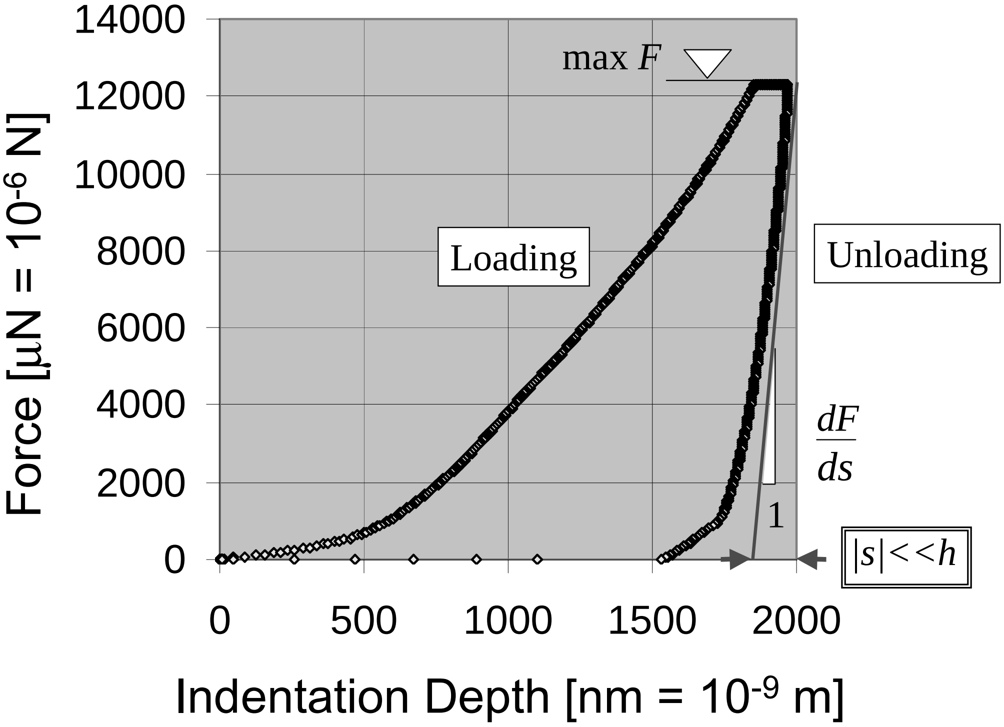


Figure 1: TOP: Conical indentation test and domains considered in the exercise. BOTTOM: Indentation Test Results: Loading and Unloading. In this exercise we are interested only in the linear elastic unloading part of the problem.

0.1 Kinematically Admissible Displacement Field

A kinematically admissible displacement field is a displacement field which satisfies the displacment boundary conditions on $\partial\Omega_{\boldsymbol{\xi}^d}$. For a rigid conical indenter, the displacement boundary is situated (a) along the indenter-material interface, along which:

$$\text{on } \partial\Omega_{\boldsymbol{\xi}^d}\left(z = h\left(1 - r/R\right); r \in [0, R]\right) : \boldsymbol{\xi}^d = s\mathbf{e}_z$$

and (b) at the boundary of the considered bulk domain, where $\boldsymbol{\xi}^d = 0$, in particular:

$$\text{on } \partial\Omega_{\boldsymbol{\xi}^d}\left(z = L - h\left(1 - r/R\right); r \in [0, R]\right) : \boldsymbol{\xi}^d = 0$$

The given displacement field,

$$\begin{aligned}
\boldsymbol{\xi}_1' &= s\left(1 - \frac{z - h\left(1 - r/R\right)}{L}\right)\mathbf{e}_z = f_1\left(z, r\right)\mathbf{e}_z \\
\boldsymbol{\xi}_2' &= s\left(1 - z/L\right)\left(1 - \frac{r - R}{L}\right)\mathbf{e}_z = f_2\left(z, r\right)\mathbf{e}_z
\end{aligned}$$

satisfies these boundary condition, i.e.

$$\begin{aligned}
\boldsymbol{\xi}_1'\left(z = h\left(1 - r/R\right); r \in [0, R]\right) &= s\mathbf{e}_z \\
\boldsymbol{\xi}_1'\left(z = L - h\left(1 - r/R\right); r \in [0, R]\right) &= 0 \\
\boldsymbol{\xi}_2'\left(z = L\right) &= 0 \\
\boldsymbol{\xi}_2'\left(r = R + L\right) &= 0
\end{aligned}$$

Furthermore, the displacement field is continuous at $r = R$:

$$\boldsymbol{\xi}_2'\left(r = R\right) - \boldsymbol{\xi}_1'\left(r = R\right) = 0$$

That is, the chosen displacement field $\boldsymbol{\xi}'$ is kinematically admissible.

For a displacement field of the form $\boldsymbol{\xi}' = f_z\left(z, r\right)\mathbf{e}_z$, the non-zero strains are:

$$\begin{aligned}
\varepsilon_{zz}' &= \frac{\partial f_z\left(z, r\right)}{\partial z} \\
\varepsilon_{zr} &= \frac{1}{2}\frac{\partial f_z\left(z, r\right)}{\partial r} = \frac{1}{2}\theta\left(\mathbf{e}_z, \mathbf{e}_r\right)\ [\text{half} - \text{distortion}]
\end{aligned}$$

Thus the strains and the strain invariants read:

- In domain Ω_1:

$$\varepsilon_{zz}' = -\frac{s}{L};\ \varepsilon_{zr}' = \frac{1}{2}\frac{\partial}{\partial r}s\left[1 - \frac{z}{L} + \frac{h}{L} - \frac{h}{L}\left(\frac{r}{R}\right)\right] = -\frac{sh}{2LR}$$

$$\text{tr}\boldsymbol{\varepsilon}' = -\frac{s}{L};\ \text{tr}\left(\boldsymbol{\varepsilon}' \cdot \boldsymbol{\varepsilon}'\right) = \left(\frac{s}{L}\right)^2\left(1 + 2\left(\frac{h}{2R}\right)^2\right) = \left(\frac{s}{L}\right)^2\left(1 + \frac{1}{2}\left(\frac{h}{R}\right)^2\right)$$

- In domain Ω_2:

$$\begin{aligned} \varepsilon'_{zz} &= \frac{\partial}{\partial z} s\left(1-\frac{z}{L}\right)\left(1-\frac{r-R}{L}\right) = -\frac{s}{L}\left(1-\frac{r-R}{L}\right) \\ \varepsilon'_{zr} &= \frac{1}{2}\frac{\partial}{\partial z} s\left(1-\frac{z}{L}\right)\left(1-\frac{r}{L}+\frac{R}{L}\right) = -\frac{s}{2L}\left(1-\frac{z}{L}\right) \end{aligned}$$

$$\mathrm{tr}\boldsymbol{\varepsilon}' = -\frac{s}{L}\left(1-\frac{r-R}{L}\right); \mathrm{tr}\left(\boldsymbol{\varepsilon}'\cdot\boldsymbol{\varepsilon}'\right) = \left(\frac{s}{L}\right)^2\left[\left(1-\frac{r-R}{L}\right)^2+\frac{1}{2}\left(1-\frac{z}{L}\right)^2\right]$$

Note that ε'_{zz} is continuous over the surface $r=R$, while ε'_{zr} is not. The associated stress, $\sigma'_{zr}=2G\varepsilon'_{zr}$ is not continuous over $r=R$, as it would be required if $\boldsymbol{\xi}'$ was the solution of the elasticity problem (i.e. $\mathbf{T}(\mathbf{n}=\mathbf{e}_r)+\mathbf{T}(\mathbf{n}=-\mathbf{e}_r)=0 \Longleftrightarrow [[\sigma_{zr}]]=0$). This shows that $\boldsymbol{\xi}'$ is not the solution of the problem, but just an approximation, appropriate to be used in an upper bound solution procedure.

0.2 Potential Energy

We first need to evaluate the potential energy of the solution, which is conveniently evaluated from Clapeyron's formula:

$$-\mathcal{E}_{com}(\boldsymbol{\sigma}) = \mathcal{E}_{pot}(\boldsymbol{\xi}) = \frac{1}{2}\left[\Phi^*(\boldsymbol{\sigma})-\Phi(\boldsymbol{\xi})\right]$$

In this displacement driven test, $\Phi(\boldsymbol{\xi})=0$; and the only contribution to the solution potential energy results from the displacement boundary condition:

$$-\mathcal{E}_{com}(\boldsymbol{\sigma}) = \mathcal{E}_{pot}(\boldsymbol{\xi}) = \frac{1}{2}\Phi^*(\boldsymbol{\sigma}) = \frac{1}{2}\int_{\partial\Omega_{\xi^d}} \boldsymbol{\xi}^d\cdot\mathbf{T}(\mathbf{n})\,da = \frac{1}{2}Fs$$

Next, we will address the potential energy associated with $\boldsymbol{\xi}'$:

$$\mathcal{E}_{pot}(\boldsymbol{\xi}) \le \mathcal{E}_{pot}(\boldsymbol{\xi}') = W(\boldsymbol{\xi}') - \Phi(\boldsymbol{\xi}')$$

In the same way as for the solution, $\Phi(\boldsymbol{\xi}')=0$ in this displacement driven test so that the potential energy can be entirely evaluated from the internal free energy in Ω_1 and Ω_2:

$$\mathcal{E}_{pot}(\boldsymbol{\xi}') = W_{\Omega_1}(\boldsymbol{\xi}'_1) + W_{\Omega_2}(\boldsymbol{\xi}'_2)$$

Thus,

- In domain Ω_1:

$$W_{\Omega_1}(\boldsymbol{\xi}'_1) = \int_{\Omega_1}\left[\frac{\lambda}{2}\left(\mathrm{tr}\boldsymbol{\varepsilon}'\right)^2 + G\,\mathrm{tr}\left(\boldsymbol{\varepsilon}'\cdot\boldsymbol{\varepsilon}'\right)\right]d\Omega = \Omega_1\left(\frac{s}{L}\right)^2\left[\frac{\lambda}{2}+G\left(1+\frac{1}{2}\left(\frac{h}{R}\right)^2\right)\right]$$

where Ω_1 is

$$\Omega_1 = 2\pi\int_{h(1-r/R)}^{L+h(1-r/R)} dz\int r\,dr = \pi L R^2$$

- In domain Ω_2:

$$\begin{aligned}
W_{\Omega_2}\left(\boldsymbol{\xi}_2'\right) &= \left(\frac{s}{L}\right)^2 \int_{\Omega_2} \left[\frac{\lambda}{2}\left(1-\frac{r-R}{L}\right)^2 + G\left(\left(1-\frac{r-R}{L}\right)^2 + \frac{1}{2}\left(1-\frac{z}{L}\right)^2\right)\right] d\Omega \\
&= 2\pi\left(\frac{s}{L}\right)^2 \int_{z=0}^{z=L}\int_{r=R}^{r=R+L} \left[\left(\frac{\lambda}{2}+G\right)\left(1-\frac{r-R}{L}\right)^2 + \frac{G}{2}\left(1-\frac{z}{L}\right)^2\right] r dr dz \\
&= 2\pi\left(\frac{s}{L}\right)^2 \left[\left(\frac{\lambda}{2}+G\right)\frac{L}{12}L\left(4R+L\right) + \frac{G}{2}\frac{1}{3}L\left(\frac{1}{2}L^2+LR\right)\right] \\
&= 2\pi\left(\frac{s}{L}\right)^2 L^3\left[\left(\frac{\lambda}{2}+G\right)\frac{1}{12}\left(4\frac{R}{L}+1\right) + \frac{G}{6}\left(\frac{1}{2}+\frac{R}{L}\right)\right]
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathcal{E}_{pot}\left(\boldsymbol{\xi}'\right) &= \pi L R^2\left(\frac{s}{L}\right)^2\left[\frac{\lambda}{2} + G\left(1+\frac{1}{2}\left(\frac{h}{R}\right)^2\right) + \left(\frac{\lambda}{2}+G\right)\frac{1}{6}\left(4+\frac{L}{R}\right) + \frac{G}{3}\left(\frac{1}{2}\frac{L}{R}+1\right)\right] \\
&= \pi L R^2\left(\frac{s}{L}\right)^2\left[\frac{1}{12}\left(10\lambda + \lambda\left(L/R\right) + 24G + 6G\left(h/R\right)^2 + 4G\left(L/R\right)\right)\right] \\
&= \pi L R^2\left(\frac{s}{L}\right)^2 \frac{1}{12}\left[\lambda\left(10+\left(L/R\right)\right) + 2G\left(12+3\left(h/R\right)^2+2\left(L/R\right)\right)\right]
\end{aligned}$$

This potential energy is greater or equal than the solution potential energy:

$$\begin{aligned}
\mathcal{E}_{pot}\left(\boldsymbol{\xi}\right) &\leq \mathcal{E}_{pot}\left(\boldsymbol{\xi}'\right) \\
\frac{1}{2}Fs &= \frac{1}{2}\frac{dF}{ds}s^2 \leq \pi L R^2\left(\frac{s}{L}\right)^2\frac{1}{12}\left[\lambda\left(10+\left(L/R\right)\right) + 2G\left(12+3\left(h/R\right)^2+2\left(L/R\right)\right)\right]
\end{aligned}$$

The previous inequality allows us to determine an upper bound for the indentation stiffness,

$$\frac{dF}{ds} \leq \left(\frac{dF}{ds}\right)^+ = \pi R\frac{1}{6}\left[\lambda\left(10\left(\frac{R}{L}\right)+1\right) + 2G\left(12\left(\frac{R}{L}\right)+3\left(\frac{R}{L}\right)\left(h/R\right)^2+2\right)\right]$$

Remarks:

3. If we use the solution,

$$\frac{dF}{ds} = \frac{2}{\sqrt{\pi}}\sqrt{A}M = 2RM$$

   we obtain the sought upper bound of the indentation stiffness:

$$M \leq M^+ = \frac{\pi}{12}\left[\lambda\left(10\left(\frac{R}{L}\right)+1\right) + 2G\left(12\left(\frac{R}{L}\right)+3\left(\frac{R}{L}\right)\left(h/R\right)^2+2\right)\right]$$

4. Using $M = E/\left(1-\nu^2\right)$, we obtain an upper bound for L/R:

$$\frac{L}{R} \leq \frac{\pi\left(14\nu^2-26\nu+12+3\left(h/R\right)^2\left(2\nu^2-3\nu+1\right)\right)}{12-2\pi-3\pi\nu^2-24\nu+5\pi\nu}$$

For $\nu = 0$, we have:

$$\frac{L}{R} \leq \frac{\pi \left(12 + 3\frac{h^2}{R^2}\right)}{12 - 2\pi}$$

That is for $h/R = 0...1$, $L/R \leq 6.6...8.2$. It is also useful to note that $L/R \to \infty$ for $\nu = 0.454\,57$ (which clearly shows some limitations of the solution).

5. It would now be appropriate to complement the upper bound approach with a lower bound solution.