2.003J/1.053J Dynamics and Control I, Spring 2007
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Lecture 3

Dynamics of a Single Particle: Angular Momentum

Example 2: Particle on String Pulled Through Hole

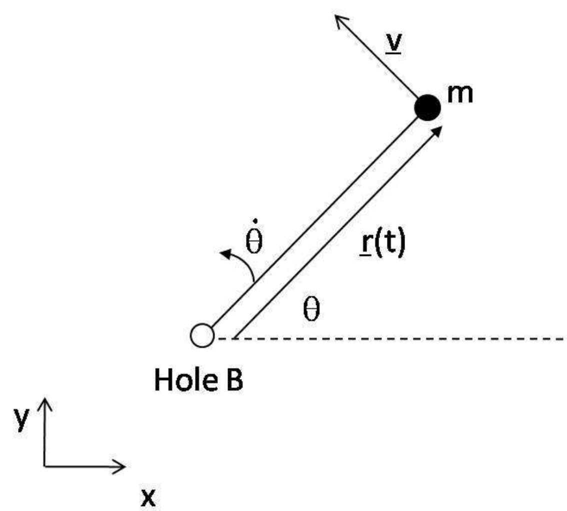

Figure 1: Particle on string pulled through hole. Tabletop with hole B. A string comes out with an attached mass. The particle is traveling around with an angular velocity $\dot{\theta}$. Figure by MIT OCW.

Assume: Frictionless surface. Inextensible String.

Pull string through hole at B such that:

$$r(t_0) = L \qquad \frac{dr}{dt}(t_0) = 0$$
$$r(t_1) = L/2 \qquad \frac{dr}{dt}(t_1) = 0$$

If $\dot{\theta}(t_0) = \dot{\theta}_0$, what is $\dot{\theta}(t_1) = \dot{\theta}_1$?

Discussion

If we use linear momentum, will need to describe forces between m and string. Thinking about angular momentum about the point B:

$$\underline{\tau}_B = \dot{\underline{h}}_B + \underline{v}_B \times m\underline{v} \leftarrow \text{ Angular momentum principle}$$
$$\underline{h}_B = \underline{r} \times m\underline{v} = \text{ Angular Momentum}$$

Now:

$$\underline{\tau}_B = \underline{r} \times \underline{F} \leftarrow \text{ Forces acting on particle} \tau_B = 0 \text{ because } \underline{r} \parallel \underline{F}$$

$$\tau_B = \dot{\underline{h}}_B + \underline{v}_B \times m\underline{v} \Rightarrow \text{ Angular momentum about } B \text{ is constant } \dot{\underline{h}}_B = \underline{0}.$$

$$\underline{\tau}_B = 0 \text{ (from above)}$$
$$\underline{v}_B = 0 \text{ because B is not moving}$$
$$\therefore h_B = \text{ Constant}$$

In Cartesian Coordinates

$$\underline{r} = r\cos\theta\hat{\imath} + r\sin\theta\hat{\jmath}$$
$$\underline{p} = m\underline{v} = m\dot{\underline{r}} = -mr\dot{\theta}\sin\theta\hat{\imath} + mr\dot{\theta}\cos\theta\hat{\jmath}$$

a. $\underline{h}_B(t_0) = \underline{r} \times \underline{p} = LmL\dot{\theta}_0\hat{k}$($\hat{k}$ is unit vector in z-direction: out of page).
b. $\underline{h}_B(t_1) = \frac{L}{2}m\frac{L}{2}\dot{\theta}_1\hat{k}$

Setting (a) = (b): $\dot{\theta}_1 = 4\dot{\theta}_0$, and velocity of particle $v_1 = 2v_0 = \frac{L}{2}4\dot{\theta}_0 = 2L\dot{\theta}_0$.

Energy is not conserved: why? The pulling force (tension) does work.

Dynamics of systems of particles

Forces on each particle may be composed as follows

$$\underline{F}_i = \underline{F}_i^{ext} + \underline{F}_i^{int}$$

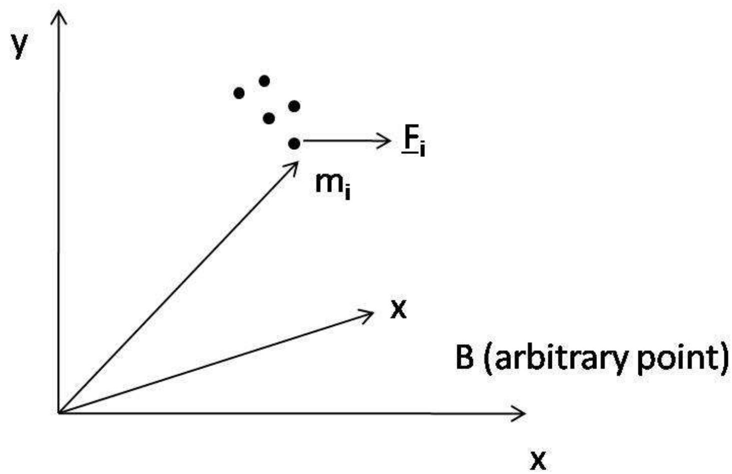

Figure 2: Dynamics of systems of particles. Figure by MIT OCW.

F_i: Resultant force acting on m_i
F_i^{ext}: External forces (e.g. gravity)
F_i^{int}: Internal forces between particles (e.g. charge attraction)

$$\underline{F}_i^{int} = \sum_{j=1}^{n} \underline{f}_{ij} \text{Force on particle i due to particle j}$$

Newton's Third Law

$$\underline{f}_{ij} = -\underline{f}_{ji}$$

Thus:

$$\sum_{i=1}^{n} \underline{F}_i^{int} = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \underline{f}_{ij} = 0$$

Sum of all internal forces is zero, therefore:

$$\sum_{i=1}^{n} F_i^{int} = 0$$

Total internal torques is also zero: demonstrate by considering an arbitrary pair of particles:

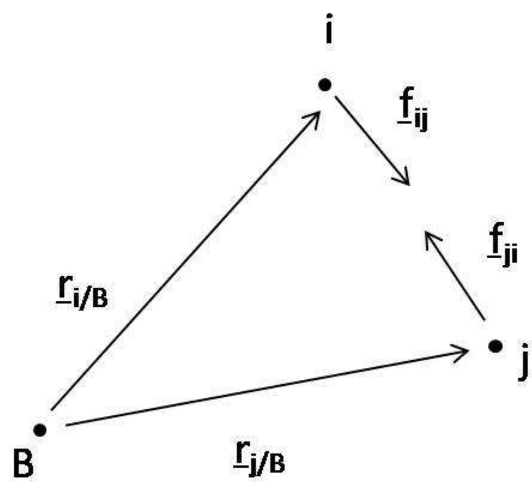

Figure 3: Arbitrary pair of particles subject to individual forces. Figure by MIT OCW.

$$\underline{\tau}_B = \underline{r}_{i/B} \times \underline{f}_{ij} + \underline{r}_{j/B} \times \underline{f}_{ji} = (\underline{r}_{i/B} - \underline{r}_{j/B}) \times \underline{f}_{ij}$$

$$\text{but } (\underline{r}_{i/B} - \underline{r}_{j/B} \parallel \underline{f}_{ij})$$

$$\therefore \underline{\tau}_B^{int} = 0 \text{ No net internal torque}$$

Center of mass

$$\underline{r}_c = \frac{\sum_{i=1}^n m_i \underline{r}_i}{\sum_{i=1}^n m_i} = \frac{\sum_{i=1}^n m_i \underline{r}_i}{M}$$

M: Total Mass of System

Note that this relation can also be written as $\sum_{i=1}^n m_i(\underline{r}_i - \underline{r}_c) = 0$ i.e. center of mass is the point about which the total mass moment is zero.

Newton's Laws for Systems of Particles

(Williams: C-1 to C-3.6)
Derivation needed to prevent mistakes in applying the laws later. Will be able to use results for rigid bodies.

Linear Momentum Principle (for a single particle)

$$\underline{F}_i = \frac{d}{dt}\underline{p}_i$$

$\underline{F}_i$: Total Force on particle i
$\underline{p}_i$: Linear momentum of particle i

$$\sum_{i=1}^{n} \underline{F}_i^{ext} + \sum_{i=1}^{n} \underline{F}_i^{int} = \frac{d}{dt}\sum_{i=1}^{n} \underline{p}_i = \frac{d}{dt}\underline{p}$$

$\underline{F}_i^{int} = 0$

$$\underline{F}^{ext} = \frac{d}{dt}\underline{p}$$

$\underline{F}^{ext}$: Sum of F external for whole system.
Note that total linear momentum:

$$\underline{p} = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} m_i v_i = M\underline{v}_c \text{ where } \underline{v}_c = \dot{r}_c = \frac{d}{dt}\sum_{i=1}^{n}\frac{m_i r_i}{M}$$

If $\sum_{i=1}^{n} \underline{F}_i^{ext} = \underline{F}^{ext} = 0 \Rightarrow \underline{p}$ =constant; therefore, $\underline{v}_c$ =constant.

Example: You have a ball as a ice skater. Throw object, both ball and skater move, but center of mass stays the same, does not move.

Angular Momentum Principle

From Newton II $\underline{F}_i = \frac{d}{dt}p_i$

Torque:

$$\underline{r}_i^{'} \times \underline{F}_i = \underline{r}_i^{'} \times \frac{d}{dt}\underline{p}_i$$

Sum over all particles.

$$\sum_{i=1}^{n} \underline{r}^{'} \times \underline{F}_i^{ext} = \sum_{i=1} \underline{r}_i^{'} \times \frac{d}{dt}\underline{p}_i$$

Later will need vectors to center of mass.

$$\sum_{i=1}^{n} \underline{\tau}_{i_B}^{ext} = \text{Sum of all external torques about B}$$

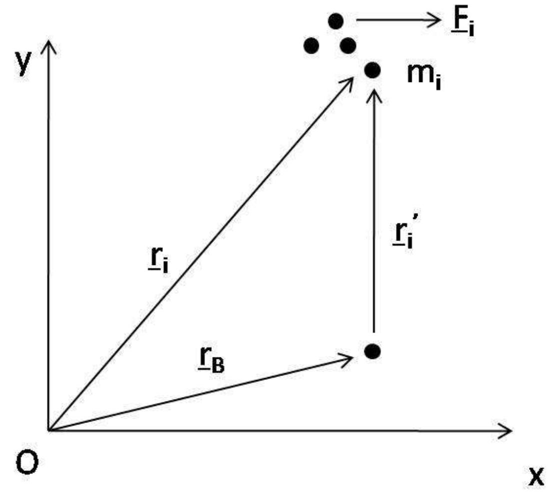

Figure 4: A system of particles subject to a force. Figure by MIT OCW.

$$\sum_{i=1}^{n} \underline{\tau}_{i_B}^{ext} = \sum_{i=1}^{n} \underline{r}^{'} \times \frac{d}{dt}\underline{p}_i = \sum_{i=1}^{n} \frac{d}{dt}(\underline{r}_i^{'} \times \underline{p}_i) - \sum_{i=1}^{n} (\frac{d}{dt}\underline{r}_i^{'})\underline{p}_i$$

$$\sum_{i=1}^{n} \underline{\tau}_{i_B}^{ext} = \frac{d}{dt}\sum_{i=1}^{n} \underline{h}_{i_B} - \sum_{i=1}^{n} \frac{d}{dt}(\underline{r}_i - \underline{r}_B) \times \underline{p}_i$$

$$\tau_B^{ext} = \frac{d}{dt}\underline{H}_B - \sum_{i=1}^{n} \underline{v}_i \times \underline{p}_i + \sum_{i=1}^{n} \underline{v}_B \times \underline{p}_i$$

$\underline{v}_B$ is the same for each $\underline{p}_i$.

$$\sum_{i=1}^{n} \underline{v}_B \times \underline{p}_i = \underline{v}_B \times \sum_{i=1}^{n} \underline{p}_i = \underline{v}_B \times \underline{p}$$

So, finally we have:

$$\underline{\tau}_B^{ext} = \frac{d}{dt}\underline{H}_B + \underline{v}_B \times \underline{P}$$

$\underline{\tau}_B^{ext}$: Total External Torque
$\frac{d}{dt}\underline{H}_B$: Total Angular Momentum
$\underline{v}_B \times \underline{p}$: Total Linear Momentum

Next time: Consequences of this expression and work-energy principle.