Massachusetts Institute of Technology
Department of Mechanical Engineering

2.003J/1.053J Dynamics & Control I

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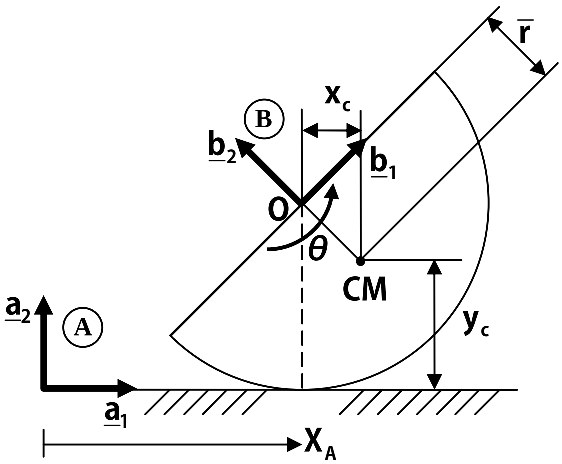
Homework 7 Solution

Problem 7.1 : Derivation of the equation of the motion for a rolling half-disk

i)



Rolling half-disk has only one degree of freedom with the constraint of ${}^{A}\underline{v}^{O} = r\theta$.
Generalized coordinate q_1 is the rotation angle of half-disk ($q_1 = \theta$).
Frame A is attached to the ground, and frame B is attached to the center of disk, O.
For kinetic energy of rolling half-disk,

$$T = \frac{1}{2} m v_{CM}^2 + \frac{1}{2} I_{CM}\omega^2 = \frac{1}{2} m \left| v_{CM} \right|^2 + \frac{1}{2} I_{CM} \dot{q}_1^2$$

To obtain the speed of half-disk at the center of mass with respect to frame A, $\left| {}^{A}\underline{v}^{CM} \right|$,

the linear velocity for rolling half-disk at the center of mass with respect to frame A, ${}^{A}\underline{v}^{CM}$ should be calculated first.

$$
\begin{aligned}
{}^A\underline{v}^{CM} &= {}^A\underline{v}^O + {}^A\underline{\omega}^B \times \underline{r}_{OCM} \\
&= -r\dot{q}_1\underline{\mathbf{a}}_1 + \dot{q}_1\underline{\mathbf{b}}_3 \times \left(-\bar{r}\underline{\mathbf{b}}_2\right) \\
&= -r\dot{q}_1\underline{\mathbf{a}}_1 + \bar{r}\dot{q}_1\underline{\mathbf{b}}_1
\end{aligned}
$$

So, the speed of half-disk at the center of mass with respect to frame A is

$$
\begin{aligned}
\left|{}^A\underline{v}^{CM}\right|^2 &= \left|-r\dot{q}_1\underline{\mathbf{a}}_1 + \bar{r}\dot{q}_1\underline{\mathbf{b}}_1\right|^2 \\
&= r^2\dot{q}_1^2 + \bar{r}^2\dot{q}_1^2 - 2r\bar{r}\dot{q}_1^2\cos q_1
\end{aligned}
\text{, since } \mathbf{a_1}\cdot\mathbf{b_1} = \cos q_1
$$

In addition, the moment of inertia around the center of mass is

$$
I_O = I_{CM} + m\bar{r}^2 = \frac{1}{2}mr^2
$$

$$
I_{CM} = \frac{1}{2}mr^2 - m\bar{r}^2
$$

Therefore,

$$
\begin{aligned}
T &= \frac{1}{2}m\left(r^2\dot{q}_1^2 + \bar{r}^2\dot{q}_1^2 - 2r\bar{r}\dot{q}_1^2\cos q_1\right) + \frac{1}{2}\left(\frac{1}{2}mr^2 - m\bar{r}^2\right)\dot{q}_1^2 \\
&= \frac{1}{2}m\left(\frac{3}{2}r^2\dot{q}_1^2 - 2r\bar{r}\dot{q}_1^2\cos q_1\right) = \frac{1}{2}m\left(\frac{3}{2}r^2 - 2r\bar{r}\cos q_1\right)\dot{q}_1^2
\end{aligned}
$$

For the potential energy of rolling half-disk,

$$
V = mg\mathbf{y_c} = mg\left(P_{CM}\cdot\underline{\mathbf{a}}_\mathbf{2}\right)
$$

where ${}^A P_{CM}$ is the position of center of mass

${}^A P_{CM}$ can be calculated as below:

$$
{}^A P_{CM} = {}^A P_O + P_{OCM} = \left(\mathbf{X_A}\underline{\mathbf{a}}_1 + r\underline{\mathbf{a}}_2\right) + \left(-\bar{r}\underline{\mathbf{b}}_2\right)
$$

Therefore,

$$
\begin{aligned}
V &= mg\left(\left(\mathbf{X_A}\underline{\mathbf{a}}_1 + r\underline{\mathbf{a}}_2\right)\cdot\underline{\mathbf{a}}_\mathbf{2} + \left(-\bar{r}\underline{\mathbf{b}}_2\right)\cdot\underline{\mathbf{a}}_\mathbf{2}\right) \\
&= mg\left(r - \bar{r}\cos q_1\right)
\end{aligned}
\text{, since } \mathbf{a_2}\cdot\mathbf{b_2} = \cos q_1
$$

From Lagrangian,

$$
L \equiv T - V : \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0
$$

where L: Lagrangian, T: kinetic energy, V: potential energy, and q: generalized coordinate

$$
\begin{aligned}
L &= \frac{1}{2} m\left(\frac{3}{2} r^2 - 2r\bar{r}\cos q_1\right)\dot{q}_1^2 - mg\left(r - \bar{r}\cos q_1\right) \\
&= \frac{1}{2} m\left(\frac{3}{2} r^2 - 2r\bar{r}\cos\theta\right)\dot{\theta}^2 - mg\left(r - \bar{r}\cos\theta\right)
\end{aligned}
\quad \text{, since } \; q_1 = \theta
$$

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial \dot{\theta}} = m\left(\frac{3}{2} r^2 - 2r\bar{r}\cos\theta\right)\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\left(\frac{3}{2} r^2 - 2r\bar{r}\cos\theta\right)\ddot{\theta} + 2mr\bar{r}\dot{\theta}^2\sin\theta$$

$$\frac{\partial L}{\partial q_1} = \frac{\partial L}{\partial \theta} = r\bar{r}m\dot{\theta}^2\sin\theta - mg\bar{r}\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = 0 \quad : \quad \left(\frac{3}{2} r^2 - 2r\bar{r}\cos\theta\right)\ddot{\theta} + r\bar{r}\dot{\theta}^2\sin\theta + g\bar{r}\sin\theta = 0$$

ii)

For small θ, $\sin\theta \approx \theta$ & $\cos\theta \approx 1$

In addition, the product of θ and higher order derivatives such as $\dot{\theta}, \ddot{\theta}$ goes to zero: $\dot{\theta}^2 \approx 0$

Therefore, the linearized equation of motion for rolling half-disk,

$$\left(\frac{3}{2} r^2 - 2r\bar{r}\right)\ddot{\theta} + g\bar{r}\theta = 0$$

iii)

Homogenous solution can be obtained as follows:

$$\left(\frac{3}{2} r^2 - 2r\bar{r}\right)\frac{d^2\theta}{dt^2} + g\bar{r}\theta = 0$$

Assume that $\theta(t) = e^{at}$, $\dfrac{d^2\theta}{dt^2} = a^2 e^{at} = a^2\theta$

$$\left(\frac{3}{2} r^2 - 2r\bar{r}\right)a\theta + g\bar{r}\theta = 0$$

$$a = \pm i\sqrt{\frac{g\bar{r}}{\left(\frac{3}{2} r^2 - 2r\bar{r}\right)}}$$

Therefore,

$$\theta(t) = A\cos\left(\sqrt{\frac{g\bar{r}}{\left(\frac{3}{2}r^2 - 2r\bar{r}\right)}}t\right) + B\sin\left(\sqrt{\frac{g\bar{r}}{\left(\frac{3}{2}r^2 - 2r\bar{r}\right)}}t\right)$$

For the given initial conditions $\theta(0) = \theta_0, \dot{\theta}(0) = \dot{\theta}_0$,

$$A = \theta_0 \quad \& \quad B = \frac{\dot{\theta}_0}{\sqrt{\frac{g\bar{r}}{\left(\frac{3}{2}r^2 - 2r\bar{r}\right)}}}$$

Therefore, the solution for the linearized equation of the motion is

$$\theta(t) = \theta_0\cos\left(\sqrt{\frac{g\bar{r}}{\left(\frac{3}{2}r^2 - 2r\bar{r}\right)}}t\right) + \frac{\dot{\theta}_0}{\sqrt{\frac{g\bar{r}}{\left(\frac{3}{2}r^2 - 2r\bar{r}\right)}}}\sin\left(\sqrt{\frac{g\bar{r}}{\left(\frac{3}{2}r^2 - 2r\bar{r}\right)}}t\right)$$

Problem 7.2 : Generate simulation codes for motion for rolling half-disk

i) The same method used in the homework #6 is used: Runge-Kutta. Most procedures are identical to the one used in homework #6. Following is the m-code for the simulation of rolling half-disk.

```
function [T,Y]=RockerRK(theta0)
% Solver for rolling half-disk
% with Runge-Kutta method
% Input argument: Initial conditions
% Output arguement: time and angle matrix

% Define some constants
r=1;            % radius of disk = 1m
rc=4*r/(3*pi);  % center of gravity
% Solve equation of motion with Runge-Kutta method
% theta0: initial conditions
% time: 0 to 10 second
```

```
[T,Y]=ode45(@(t,theta) Rocker(t,theta,r,rc),[0 10],theta0);
% extract only angle matrix
Y=Y(:,1);
end

function dTHETA=Rocker(t,theta,r,rc)
% descrive equation of motion for rolling half-disk

g=9.81;        % gravity
% angular velocity
dTHETA(1,1)=theta(2);
% angular acceleration
dTHETA(2,1)=-(r*rc*theta(2)^2+g*rc)*sin(theta(1))/(3/2*r^2-
2*rc*r*cos(theta(1)));
end
```

ii) The analytic solution you obtained in P7.1 iv) is used to find the trajectory of rotation angle of rolling half-disk. First, you make time vector which have numbers from 0 to 10 with enough step to describe motion well (I chose 0.01 sec.) Then, some constants are given, and calculate solution with respect to time matrix. Matrix operation should be used. The following is m-code for calculating the analytical solution for rolling half-disk.

```
function [T,Y]=RockerAN(theta0)
% Solver for rolling half-disk
% with analytic solution
% Input argument: Initial conditions
% Output arguement: time and angle matrix

% define some constants
g=9.81;        % gravity
r=1;           % radius of disk
rc=4*r/(3*pi);  % center of gravity

% coefficients for analytic solution
Omega=sqrt((g*rc)/(3/2*r^2-2*r*rc));   % natural frequency
A=theta0(1);
```

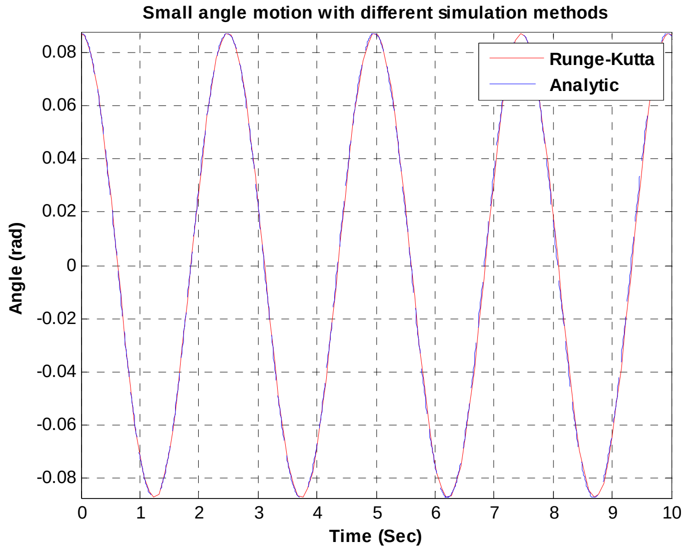
```
B=theta0(2)/Omega;

% define time series with time step of 0.01
T=[0:0.01:10]';
% calculate rotation angle at a given time
Y=A*cos(Omega*T)+B.*sin(Omega*T);
End
```

Problem 7.3 : Trajectory of $\theta(t)$ for both small and large angle oscillations

i) As expected, the results with Runge-Kutta method and analytic approach are pretty close. The linearization works for small angle rotation very well. Note that the unit of angle is radian, not degree, when you give the initial conditions to function you made. Triangular function in MATLAB such as sin, cos, and tan accept for only radian. The following codes describe how to generate the below plot where the result with different simulation methods are compared.

```
>> [T1,Y1]=RockerRK([5*pi/180,0]);
>> [T2,Y2]=RockerAN([5*pi/180,0]);
>> plot(T1,Y1,'r-',T2,Y2,'b--');
>> grid on; axis tight;
>> xlabel('\bfTime (Sec)'); ylabel('\bfAngle (rad)');
>> title('\bfSmall angle motion with different simulation methods');
>> legend('\bfRunge-Kutta','\bfAnalytic');
```



ii) For the case of large angle rotation of half-disk, the linearized rotation motion is quite different from numerical simulation of nonlinear rotation motion. Rotation obtained with nonlinear equation is a little slower than the one with the linear equation since nonlinear terms in the differential equation is dominant when the rotation angle becomes larger. The following codes describe how to generate the below plot where the result with different simulation methods are compared.

```
>> [T1,Y1]=RockerRK([30*pi/180,0]);
>> [T2,Y2]=RockerAN([30*pi/180,0]);
>> plot(T1,Y1,'r-',T2,Y2,'b--');
>> grid on; axis tight;
>> xlabel('\bfTime (Sec)'); ylabel('\bfAngle (rad)');
>> title('\bfLarge angle motion with different simulation methods');
>> legend('\bfRunge-Kutta','\bfAnalytic');
```