1.017/1.010 Class 3
Probability

Conceptual framework

Probability theory provides a conceptual framework for analyzing uncertain outcomes of experiments

Definitions

An **experiment** is defined by:

1. A set of experimental **outcomes**
2. A collection of **events** constructed from these outcomes
3. A rule which assigns **probabilities** to the events

Definition of outcomes is problem/context dependent.

Sample space S is set of all possible experimental outcomes

An **event** is a set of outcomes. An **elementary event** is a single outcome.

Probability $P(A)$ of an event A is a number assigned to the event that meets the following requirements (probability axioms):

1. $P(A) \geq 0$
2. $P(S) = 1$
3. $P(A_1 + A_2 + ... + A_N) = P(A_1) + P(A_2) + ... + P(A_N)$
   For $A_1 + A_2 = A_1 \bigcup A_2$
   $A_1 A_2 = A_1 \bigcap A_2 = 0$ (A_1, A_2 are mutually exclusive events)
   Note $A_1 + A_2$ and $A_1 A_2$ are distinct events with their own probabilities

$P(A)$ is intended to convey the **likelihood** that A occurs, on a scale from 0 to 1

Probability properties (follow from axioms)

1. $P(A_1 + A_2) = P(A_1) + P(A_2) - P(A_1 A_2)$
   for $A_1 A_2$ not mutually exclusive
2. $P(\sim A) = 1 - P(A)$; $\sim A$ = complement of A

Methods for assigning probabilities to events

Empirical / Relative frequency approach – Experiment is repeated many times. Probability of an event is number of times event is observed

divided by total number of repetitions.

Conceptual / deductive approach – Probability of an event is fraction of total outcomes in sample space associated with this event (total outcomes in sample space may be infinite).

Distributional approach – Probability of an event is derived from a specified **probability distribution** (more on this later)

Examples:

Experiment: Three successive coin tosses
Outcomes/elementary events: Different sequences of heads and tails
Sample space: 8 possible outcomes/sequences
Typical event: Set of outcomes that yield 2 heads and 1 tail
Assigning Probabilities (conceptual): $P(A)$ is fraction of total outcomes in event (3/8).
Assigning Probabilities (empirical): Replicate sequence of 3 tosses many times, $P(A)$ is observed fraction of replicates giving 2 heads and a tail (not necessarily 3/8).

Experiment: Toss of a dart onto a square region of side 2
Outcomes/elementary events: Different locations where dart can land
Sample space: Infinite number of possible locations in square
Typical events: Inscribed circle of radius 1
Assigning Probabilities (conceptual): $P(A)$ is fraction of S covered by circle (pi/4).
Assigning Probabilities (empirical): Toss dart many times, $P(A)$ is fraction of tosses falling in inscribed circle (not necessarily pi/4).

Experiment: Selection of a student from a hypothetical infinite population
Outcomes/elementary events: Height of any given student
Sample space: Infinite number of all possible heights
Typical event: Student height is between 5 and 6 feet
Assigning Probabilities (distributional): Assume $P(A)$ is partial area under a specified histogram between 5 and 6 feet.
Assigning Probabilities (empirical): $P(A)$ is fraction of large sample of student heights that fall between 5 and 6 feet.

Exercise: Virtual experiments

Write a script that repeats the first two experiments described above (3 coin tosses and 1 dart toss) 20 times. Examine the results of all simulated 20 experiments, manually count the number of times that the events specified above occur, and use the relative frequency approach to estimate the probability of the following events:

For dice toss: Event A = (2 heads and 1 tail)
For dart toss: Event A = (dart falls in inscribed circle of radius 1)

Compare the two relative frequency results with conceptual probability calculations.

Some relevant MATLAB functions: `rand, unidrnd, ceil, sum, pi`

MATLAB function: virtual.m