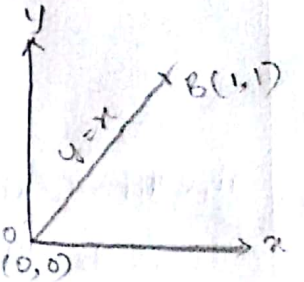
COMPLEX INTEGRATION

1. Evaluate $\int_0^{1+i} (x^2 - iy)\,dz$ along the paths (i) $y = x$ (ii) $y = x^2$.

Sol: i) Along OB whose equation is $y = x \Rightarrow dy = dx$ and x varies from 0 to 1.

$$\int_0^{1+i} (x^2 - iy)\,dz = \int_{(0,0)}^{(1,1)} (x^2 - iy)(dx + i\,dy)$$

$$\therefore \int_{OB} (x^2 - iy)\,dz = \int_{x=0}^{1} (x^2 - iy)(dx + i\,dy) = \int_{x=0}^{1} (x^2 - ix)(dx + i\,dx)$$



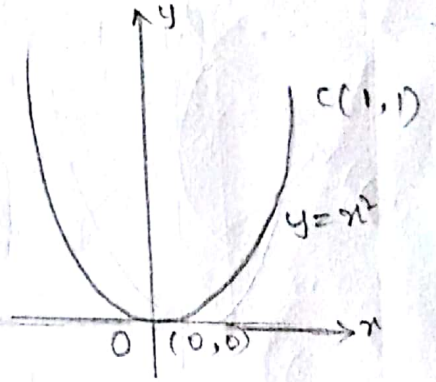
$$= (1+i)\int_0^1 (x^2 - ix)\,dx = (1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1$$

$$= (1+i)\left[\frac{1}{3} - \frac{i}{2}\right]$$

ii) Along the parabola whose equation is $y = x^2$

$\therefore dy = 2x\,dx$

Now $\int_0^{1+i} (x^2 - iy)\,dz = \int_{(0,0)}^{(1,1)} (x^2 - iy)(dx + i\,dy)$



$$\int_{OC} (x^2 - iy)\,dz = \int_{x=0}^{1} (x^2 - ix^2)(dx + i\,2x\,dx)$$

$$= (1-i)\int_{x=0}^{1} x^2(1 + 2ix)\,dx$$

$$= (1-i)\int_{x=0}^{1} (x^2 + 2ix^3)\,dx$$

$$= (1-i)\left[\frac{x^3}{3} + i\frac{x^4}{2}\right] = (1-i)\left[\frac{1}{3} + \frac{i}{2}\right]$$

2. (a) Integrate $f(z) = x^2 + ixy$ from A(1,1) to B(2,8) along.

i) The straight line AB ii) The curve C: $x = t$, $y = t^3$

(b) Integrate $f(z) = x^2 + ixy$ from A(1,1) to B(2, 4) along curve $x = t$, $y = t^2$.

Sol: i) $\int f(z)\,dz = \int_{(1,1)}^{(2,8)} (x^2 + ixy)(dx + i\,dy)$

Along AB: Equation of AB passing through A(1,1) and B(2,8) is

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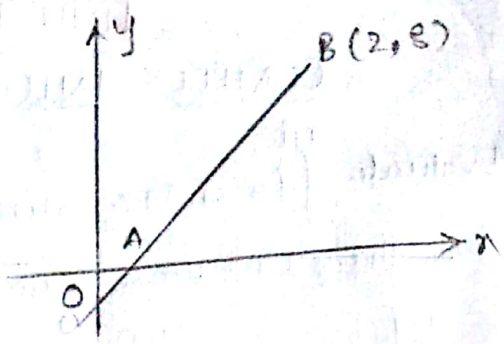
$$\frac{y-1}{8-1} = \frac{x-1}{2-1} \Rightarrow y = 7x-6 \Rightarrow dy = 7dx$$



$$\therefore \int_{AB} f(z)\,dz = \int_{x=1}^{2} \{x^2 + ix(7x-6)\}\{dx + 7i\,dx\}$$

$$= (7i+1)\int_{x=1}^{2} (7i+1)x^2 - 6ix\,dx$$

$$= (7i+1)\left[(7i+1)\frac{x^3}{3} - 3ix^2\right]_1^2 = (7i+1)\left[(7i+1)\frac{8}{3} - 12i - \frac{(7i+1)}{3} + 3i\right]$$

$$= (7i+1)\left[(7i+1)\cdot\frac{7}{3} - 9i\right] = \frac{7i+1}{3}[49i + 7 - 27i] = \frac{7i+1}{3}(22i+7)$$

(ii) Along C whose parametric equations are $x = t, y = t^3$

$\therefore dx = dt$, $dy = 3t^2 dt$

$A(1,1) \Rightarrow t = 1$ and $B(2,8) \Rightarrow t = 2$

$$\int_C f(z)\,dz = \int_{(1,1)}^{(2,8)} (x^2 + ixy)(dx + i\,dy)$$

$$\therefore \int_C f(z)\,dz = \int_{t=1}^{t=2} (t^2 + it^4)(dt + i3t^2\,dt) = \int_1^2 (t^2 + it^4)(1 + 3it^2)\,dt$$

$$= \int_1^2 (t^2 + it^4 + 3it^4 - 3t^6)\,dt = \int_1^2 [t^2 + (1+3i)t^4 - 3t^6]\,dt$$

$$= \left[\frac{t^3}{3} + (1+3i)\frac{t^5}{5} - \frac{3}{7}t^7\right]_1^2$$

$$= \frac{8}{3} + (1+3i)\frac{32}{5} - \frac{3}{7}(128) - \frac{1}{3} - \frac{(1+3i)}{5} + \frac{3}{7}$$

$$= \frac{1}{105}(-4818 + i1953)$$

Evaluate $\int_C (y^2 + 2xy)\,dx + (x^2 - 2xy)\,dy$ where C is the boundary of the region by $y = x^2$ and $x = y^2$.

Given curves are $y = x^2$ .... ① and $x = y^2$ → ②

The two curves (1) and (2) intersect at points (0, 0) and (1, 1)

The positive direction in traversing C is as show in figure.

Along $y = x^2$,

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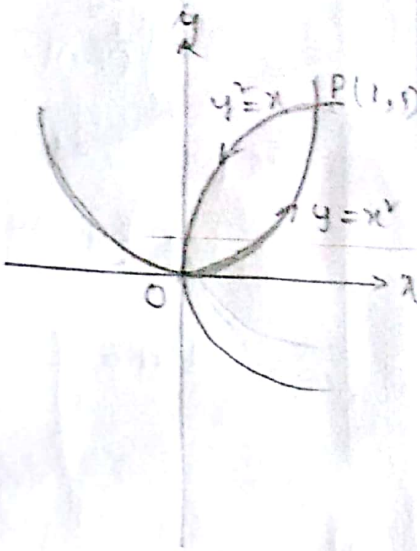
the line integral is

$$= \int_{x=0}^{1} (x^4 + 2x^3)dx + (x^2 - 2x^3)d(x^2)$$

$$= \int_0^1 (4x^3 - 3x^4)dx = \left[4\cdot\frac{x^4}{4} - 3\cdot\frac{x^5}{5}\right]_0^1$$

$$= \left[x^4 - \frac{3}{5}x^5\right]_0^1 = 1 - \frac{3}{5} = \frac{2}{5} \rightarrow (3)$$



Along $y^2 = x$,

the line integral is

$$= \int_{x=1}^{0} (x + 2x^{3/2})dx + (x^2 - 2x^{3/2})d(\sqrt{x})$$

$$\left\{(or) \int_{y=1}^{0} (y^2 + 2y^3)2y\,dy + (y^4 - 2y^3)dy\right\}$$

$$= \int_1^0 (x + 2x^{3/2})dx + (x^2 - 2x^{3/2})\frac{1}{2\sqrt{x}}dx$$

$$= \frac{5}{2}\int_1^0 x^{3/2}dx = \frac{5}{2}\left[\frac{x^{5/2}}{5/2}\right]_1^0 = 0 - 1 = -1 \rightarrow (4)$$

Hence the line integral over $C = \frac{2}{5} - 1 = \frac{-3}{5}$ [Adding (3) and (4)]

4. Evaluate (i) $\int \frac{e^{2z}}{(z-1)(z-2)}dz$ where C is the circle $|z| = 3$

(ii) $\int \frac{e^z}{(z-1)(z-2)}dz$ where C is $|z| = 2$

Sol. (i) $\frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$ using partial fractions.

$$\therefore \oint_C \frac{e^{2z}dz}{(z-2)(z-1)} = \oint_C \frac{e^{2z}}{(z-2)}dz - \oint_C \frac{e^{2z}}{(z-1)}dz$$

The points $z = 1, 2$ lies inside C.

Because e^{2z} is analytic everywhere, according to Cauchy's integral formula

$$\oint_C \frac{e^{2z}}{(z-2)}dz - \oint_C \frac{e^{2z}}{(z-1)}dz$$

$$= \left[2\pi i e^{2z}\right]_{z=2} - \left[2\pi i e^{2z}\right]_{z=1}$$

$$= 2\pi i\left[e^4 - e^2\right]$$

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ii, $\frac{1}{(z-1)(z-4)} = \frac{1}{3}\left[\frac{1}{(z-4)} - \frac{1}{(z-1)}\right]$ using partial fractions

$$\therefore \int_C \frac{e^z}{(z-1)(z-4)}\,dz = \frac{1}{3}\int_C \frac{e^z}{(z-4)}\,dz - \frac{1}{3}\int \frac{e^z}{z-1}\,dz$$

The point $z=4$ lies outside C. But point $z=1$ lies inside C.

$\therefore$ By Cauchy's theorem $\int_C \frac{e^z}{z-4}\,dz = 0$

Hence $\int_C \frac{e^z}{(z-1)(z-4)}\,dz = 0 - \frac{1}{3}\int \frac{e^z}{z-1}\,dz$

$= -\frac{1}{3}2\pi i (e^z)_{z=1}$, using Cauchy's integral theorem

$= -\frac{1}{3}2\pi i e.$

Evaluate $\int_C \frac{z+4}{z^2+2z+5}$ where C is the circle i, $|z|=1$ ii, $|z+1-i|=2$

iii, $|z+1+i|=2$

$$f(z) = \frac{z+4}{(z^2+2z+5)} = \frac{z+4}{(z+1)^2-(2i)^2} = \frac{z+4}{(z+1+2i)(z+1-2i)}$$

Hence the singularities of the function $\frac{z+4}{z^2+2z+5}$ are $-1-2i$ and $-1+2i$. So the function is not analytic at the points $(-1,-2)$ and $(-1,2)$.

i, When C is the circle $|z|=1$ i,e $x^2+y^2-1=0=f(x,y)$

Both $(-1,-2)$ and $(-1,2)$ lie outside the circle because of substituting the values of x and y in the equations of the circle, $f(x,y)>0$ in both the cases.

$\therefore f(z) = \frac{z+4}{z^2+2z+5}$ ~~$dz = 2x$~~ is analytic at all points within and on the circle $|z|=1$.

$\therefore$ By Cauchy's integral theorem, $\int \frac{z+4}{z^2+2z+5}\,dz = 0$

i, When C is the circle $|z+1-i| = 2$ i,e $(x+1)^2+(y-1)^2 = 4$

The point $(-1,-2)$ i,e $z=-1-2i$ does not lie within C whereas the point $(-1,2)$ i,e $z=-1+2i$ lies inside.

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$$\therefore \int_C \frac{z+4}{(z+1+2i)(z+1-2i)}\,dz = \int \frac{\frac{z+4}{z+1+2i}}{z+1-2i}\,dz$$

$$= \int \frac{f(z)}{(z-a)}\,dz \text{ where } f(z) = \frac{z+4}{z+1+2i} \text{ and } a = -1+2i$$

(ii) By Cauchy's integral formula $\int \frac{f(z)}{(z-a)}\,dz = 2\pi i f(a) \rightarrow (1)$

But $f(z) = \frac{z+4}{(z+1+2i)}$, $a = -1+2i$

$\therefore f(a) = f(-1+2i) = \frac{-1+2i+4}{-1+2i+1+2i} = \frac{3+2i}{4i}$

Substituting in equ (1), we get

$$\int_C \frac{z+4}{z^2+2z+5}\,dz = 2\pi i\left[\frac{3+2i}{4i}\right] = \frac{\pi}{2}(3+2i)$$

(iii) When C is the circle $|z+1+i| = 2$ i.e $(x+1)^2+(y+1)^2 = 4$

The point $(-1,-2)$ i.e $z = -1-2i$ lies inside whereas the point $z = -1+2i$ lies out-side the circle.

$$\therefore \int \frac{z+4}{(z+1+2i)(z+1-2i)}\,dz = \int \frac{\frac{z+4}{z+1-2i}}{z+1+2i}\,dz = \int_C \frac{f(z)}{(z-a)}\,dz$$

where $f(z) = \frac{z+4}{z+1-2i}$ and $a = -1-2i$

Hence $\int_C \frac{z+4}{(z+1+2i)(z+1-2i)}\,dz = 2\pi i f(a) = 2\pi i f(-1-2i)$

$$= 2\pi i\left[\frac{-1-2i+4}{-1-2i-2i+4}\right] = 2\pi i\left[\frac{3-2i}{-4i}\right] = \frac{\pi}{2}(2i-3)$$

~~Evaluate $\int \frac{z^2-z-1}{z(z-1)^2}$~~

Using Cauchy's integral formula, evaluate $\int \frac{z}{(z-1)(z-2)^2}\,dz$, where $C: |z-2| = 1/2$

The integral has two singular points at $z=1$ and $z=2$ of which only $z=2$ lies inside C.

$f(z) = \frac{z}{(z-1)}$ is analytic on and which C.

Here $a = 2$ and $n = 1$

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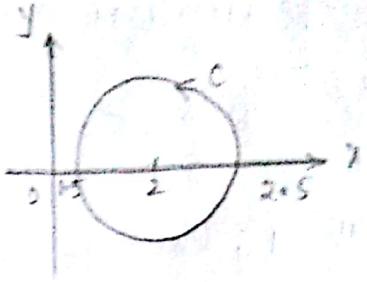
∴ by Cauchy's integral formula

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz,$$



we get,

$$\int_C \frac{z}{(z-1)(z-2)^2}\,dz = 2\pi i\left[\frac{d}{dz}\left[\frac{z}{z-1}\right]\right]_{z=2}$$

$$= 2\pi i\left[\frac{d}{dz}\left[1+\frac{1}{z-1}\right]\right]_{z=2}$$

$$= 2\pi i\left[\frac{-1}{(z-1)^2}\right]_{z=2} = -2\pi i$$

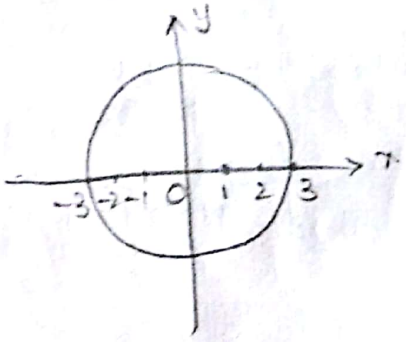
7. Use Cauchy's integral formula to evaluate $\int_C \frac{e^z}{(z+2)(z+1)^2}\,dz$ where C is the circle $|z|=3$

Singular points of integrands are given by putting the denominator equal to zero.

$(z+2)(z+1)^2 = 0 \Rightarrow z = -1, -2$

Both singular points are inside the given circle with centre at origin and radius 3

$$\therefore \int \frac{e^z}{(z+2)(z+1)^2}dz = \int \frac{e^z}{(z+2)(z+1)^2}dz + \int \frac{e^z}{(z+2)(z+1)^2}dz$$



$$= \int \frac{\frac{e^z}{z+2}}{(z+1)^2}dz + \int \frac{\frac{e^z}{(z+1)^2}}{z+2}dz$$

$$= 2\pi i f'(-1) + 2\pi i f(-2),$$

by Cauchy's integral formula

$$= 0 + 2\pi i \frac{e^{-2}}{(-2+1)^2} = \frac{2\pi i}{e^2}$$

8. Taylors Theorem Statement:-

If a function $f(z)$ is analytic inside a circle 'C' whose centre is 'a' then for all values of z inside C $f(z) = f(a) + f(z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots + \frac{(z-a)^n}{n!}f^n(a) + \cdots$

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Laurent Series Statement:- If $f(z)$ is analytic inside and on the boundary of ring shaped region R bounded by 2 concentric circles C_1 & C_2 of radius r_1 and r_2 respectively $(r_1 > r_2)$ having centre at a then for all z in 'R' $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_1(z-a)^{-1} + a_{-2}(z-a)^{-2} + \cdots$

$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw, \quad n = 0, 1, 2$$

$$a_{-n} = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw, \quad n = 1, 2, 3$$

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{Analytic part}} + \underbrace{\sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}}_{\text{principle part}}$$

9. Find Taylor's series expansion for the function $f(z) = \frac{1}{(1+z)^2}$ with centre at $-i$.

Sol:- By Taylors theorem

$$f(z) = f(a) + f(z-a)f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots + \frac{(z-a)^n}{n!} f^n(a) + \cdots$$

Put $a = -i$

$$\therefore f(z) = f(-i) + f(z+i)f'(-i) + \frac{(z+i)^2}{2!} f''(-i) + \cdots + \frac{(z+i)^n}{n!} f^n(-i) + \cdots \rightarrow (1)$$

Here $f(z) = \frac{1}{(1+z)^2}$

$$\therefore f^n(z) = (-1)^n \frac{(n+1)!}{(1+z)^{n+2}}$$

Now $f(-i) = \frac{1}{(1-i)^2} = \frac{i}{2}$ and $f^n(-i) = \frac{(-1)^n (n+1)}{(1-i)^{n+2}}$

Substituting in equ(1), we get

$$\frac{1}{(1+z)^2} = \frac{i}{2} + \sum_{n=1}^{\infty} \frac{(z+i)^n}{n!} \frac{(-1)^n (n+1)!}{(1-i)^{n+2}} = \frac{i}{2} + \sum_{n=1}^{\infty} (-1)^n (n+1) \frac{(z+i)^n}{(1-i)^{n+2}}$$

$$= \frac{i}{2} + \sum_{n=1}^{\infty} (-1)^n (n+1) \frac{(z+i)^n}{(1-i)^n} \frac{1}{(1-i)^2} = \frac{i}{2} + \sum_{n=1}^{\infty} (-1)^n (n+1) \frac{(z+i)^n}{(1-i)^n (-2i)}$$

$$= \frac{i}{2} + \frac{i}{2} \sum_{n=1}^{\infty} (-1)^n (n+1) \frac{(z+i)^n}{(1-i)^n}$$

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Obtain the Taylors series to represent the function $\frac{z^2-1}{(z+2)(z+3)}$

Let $f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$ Resolving into partial fractions

$= 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)} = 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$

Expanding by Binomial Series,

$f(z) = 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right]$

$= 1 + \frac{3}{2}\sum_{z=0}^{\infty} \frac{(-1)^n}{2^n} z^n - \frac{8}{3}\sum_{z=0}^{\infty} (-1)^n \frac{z^n}{3^n}$

$= 1 + \frac{3}{2} - = 1 + \sum_{z=0}^{n} (-1)^n \left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right] z^n$

Expand $f(z) = \frac{1}{z^3-3z+2}$ in the region (i) $0<|z-1|<1$ (ii) $1<|z|<2$

: We have

$f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)}$

$= \frac{(z-1)-(z-2)}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$

$\therefore z=1, z=2$ are the singular points of $f(z)$.

(i) The function $f(z)$ is analytic in the ring shaped region $0<r_2<|z-1|<r_1$

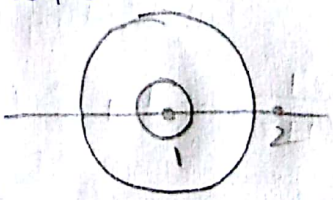
Put $z-1=w$ $\therefore z = w+1 \Rightarrow z-2 = w+1-2 = w-1$

$\therefore f(z) = \frac{1}{w-1} - \frac{1}{w} = -\frac{1}{w} - \frac{1}{1-w}$

$= -\frac{1}{w} - [1+w+w^2+\cdots]$ if $w<1$ and $w \neq 0$

$= -\frac{1}{(z-1)} - \sum_{n=0}^{\infty} (z-1)^n$ if $0<|z-1|<1$

$= -\sum_{n=-1}^{\infty} (z-1)^n$ if $0<|z-1|<1$



(ii) Given $1<|z|<2$ i.e $1<|z|$ and $|z|<2$ or $\left|\frac{1}{z}\right|<1$ and $\left|\frac{z}{2}\right|<1$

$\therefore f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{-2\left[1-\frac{z}{2}\right]} - \frac{1}{z\left[1-\frac{1}{z}\right]} = -\frac{1}{2}\left[1-\frac{z}{2}\right]^{-1} - \frac{1}{z}\left[1-\frac{1}{z}\right]$

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$= -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]$

$= -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right] - \left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]$

Expand $\frac{z-2}{(z+1)z(z-2)}$ about the point $z=-1$ in the region $1<|z+1|<3$

by Laurent's series.

Let $f(z) = \frac{z-2}{(z+1)(z)(z-2)}$

put $z+1=w$, then $z=w-1$

$\therefore f(z) = \frac{(w-1)-2}{w(w-1)(w-1-2)} = \frac{w-9}{w(w-1)(w-3)}$

By partial fractions,

$f(z) = \frac{A}{w} + \frac{B}{w-1} + \frac{C}{w-3} = \frac{3}{w} + \frac{1}{w-1} + \frac{2}{w-3}$

$= -\frac{3}{w} + \frac{1}{w\left[1-\frac{1}{w}\right]} - \frac{2}{3\left[1-\frac{w}{3}\right]} = -\frac{3}{w} + \frac{1}{w}\left[1-\frac{1}{w}\right]^{-1} - \frac{2}{3}\left[1-\frac{w}{3}\right]^{-1}$

$= -\frac{3}{w} + \frac{1}{w}\left[1+\frac{1}{w}+\frac{1}{w^2}+\cdots\right] - \frac{2}{3}\left[1+\frac{w}{3}+\frac{w^2}{3^2}+\cdots\right]$

$= \left[-\frac{2}{w}+\frac{1}{w^2}+\frac{1}{w^3}+\cdots\right] - \frac{2}{3}\left[1+\frac{w}{3}+\frac{w^2}{3^2}+\cdots\right]$

$= \frac{-2}{(z+1)} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3}\cdots - \frac{2}{3}\left[1+\frac{(z+1)}{3}+\frac{(z+1)^2}{3^2}+\cdots\right]$

The above series is valid in the region $\left|\frac{1}{w}\right|<1$ and $\left|\frac{w}{3}\right|<1$

i.e $1<|w|$ and $|w|<3$ i.e $1<|w|<3$ or $1<|z+1|<3$.

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1. Find the poles and residues of $\frac{1}{z^2-1}$.

Sol: Let $f(z) = \frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$, using partial fractions. poles of $f(z)$ are given by $(z-1)(z+1)=0$ i.e $z = \pm 1$. These are simple poles. To find the residue at $z=1$, we expand the function in a Laurent series in power of $z-1$.

To expand in powers of $z-1$, we write

$$f(z) = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z-1+2}\right) = \frac{1}{2}\left(\frac{1}{u} - \frac{1}{u+2}\right) \quad \text{where } u = z-1$$

$$= \frac{1}{2u} - \frac{1}{4}\left(\frac{1}{1+\frac{u}{2}}\right) = \frac{1}{2u} - \frac{1}{4}\left(1+\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{2u} - \frac{1}{4}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \dots\right)$$

$$= \frac{1}{2u} - \frac{1}{4} + \frac{u}{8} - \frac{u^2}{16} + \frac{u^3}{32} - \dots$$

$$= \frac{-1}{4} + \frac{1}{2(z-1)} + \frac{z-1}{8} - \frac{(z-1)^2}{16} + \dots$$

Coefficient of $\frac{1}{z-1} = \frac{1}{2}$ and hence residue at $z=1$ is $\frac{1}{2}$

To find the residue at $z=-1$, we expand $f(z)$ in a Laurent series in powers of $z+1$.

we have $f(z) = \frac{1}{2}\left(\frac{1}{z+1-2} - \frac{1}{z+1}\right) = \frac{1}{2}\left(\frac{1}{u-2} - \frac{1}{u}\right)$ where $u = z+1$

$$= \frac{-1}{4}\left(\frac{1}{1-\frac{u}{2}}\right) - \frac{1}{2u} = \frac{-1}{4}\left(1-\frac{u}{2}\right)^{-1} - \frac{1}{2u}$$

$$= \frac{-1}{4}\left(1 + \frac{u}{2} + \frac{u^2}{4} + \frac{u^3}{8} + \dots\right) - \frac{1}{2u}$$

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$$= -\frac{1}{4}\left[1 + \frac{z+1}{2} + \frac{(z+1)^2}{4} + \frac{(z+1)^3}{8} + \dots\right] - \frac{1}{2(z+1)}$$

Thus coefficient of $\frac{1}{z+1} = -\frac{1}{2}$ $\therefore$ Residue at $z=1$ is $-\frac{1}{2}$.

2.) Find the residue of $\frac{1}{(z-\sin z)}$ at $z=0$.

Solution: Let $f(z) = \frac{1}{z-\sin z}$

The function $f(z)$ has a laurent's expansion.

$$f(z) = \frac{1}{z - \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots\right]} = \frac{1}{\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \dots}$$

$$= \frac{1}{\frac{z^3}{3!}\left(1 - \frac{z^2}{20} + \frac{z^4}{840} - \dots\right)} = \frac{6}{z^3}\left[1 - \left(\frac{z^2}{20} - \frac{z^4}{840} + \dots\right)^{-1}\right]$$

$$= \frac{6}{z^3}\left[1 + \left(\frac{z^2}{20} - \frac{z^4}{840}\right)^2 + \dots\right] = \frac{6}{z^3}\left[1 + \frac{z^2}{20} - \frac{z^4}{840} + \frac{z^4}{400} - \dots\right]$$

$$= \frac{6}{z^3}\left[1 + \frac{z^2}{20} + \frac{20z^4}{840} + \dots\right]$$

$$= \frac{6}{z^3} + \frac{6}{20}\cdot\frac{1}{z} + \frac{120z}{840} + \dots$$

$\therefore$ Residue at the pole $z=0$ is coefficient of $\frac{1}{z} = \frac{6}{20} = \frac{3}{10}$

3.) Determine the poles of the function

(i) $f(z) = \frac{z^2}{(z-1)^2(z+2)}$

(ii) $f(z) = \frac{z^2}{(z+1)^2(z+2)}$ and the residues at each pole.

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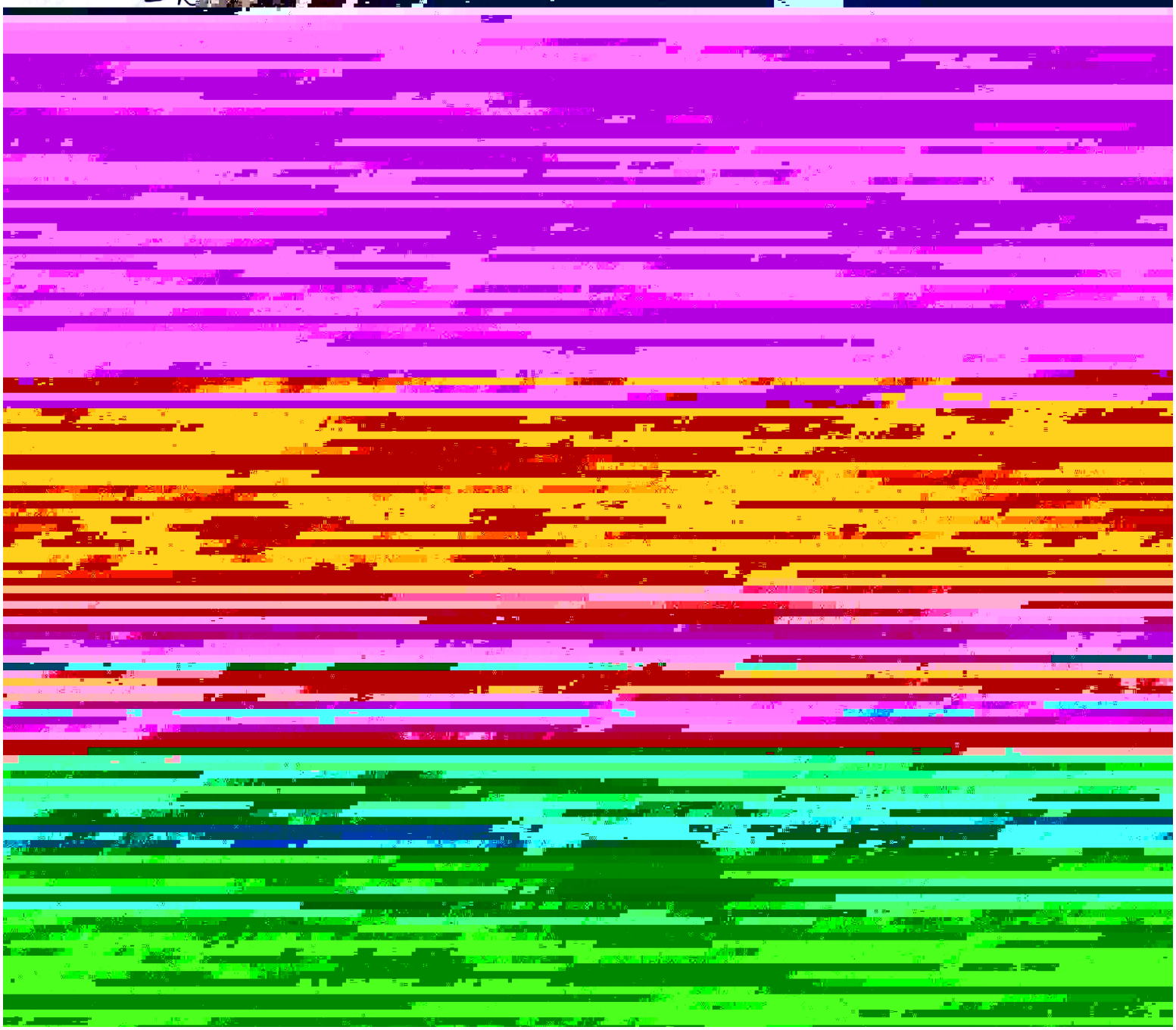
Solution: $z=1$ and $z=-2$ are the zeroes of denominator of order 2 and 1 respectively. $\therefore z=1$ is a pole of order 2 and $z=-2$ is a pole of order 1 of $f(z)$.

(i) $\left[\text{Res } f(z)\right]_{z=-2} = \lim_{z\to -2} (z+2) f(z) = \lim_{z\to -2} (z+2) \dfrac{z^2}{(z-1)^2(z+2)}$

$\lim_{z\to -2} \dfrac{z^2}{(z-1)^2} = \dfrac{4}{9}$

$\left[\text{Res } f(z)\right]_{z=1} = \lim_{z\to 1} \dfrac{1}{1!} \dfrac{d}{dz} \left[(z-1)^2 f(z)\right]$

$= \lim \left[\dfrac{d}{dz} \left(\dfrac{z^2}{\quad} \right) \right] = \lim \left[\dfrac{d}{dz} \left(\dfrac{z^2}{\quad} \right) \right]$



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• Calculation of Residues

(i) when $z = z_0$ is a simple pole.

In this case the laurent's series expansion becomes

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{a_{-1}}{z-z_0}$$

$\therefore \lim_{z \to z_0} (z-z_0) f(z) = a_{-1}$

$\therefore$ Res $(f; z = z_0)$ or $[\text{Res } f(z)]_{z=z_0} = \lim_{z \to z_0} (z-z_0) f(z) = a_{-1}$

$$= \frac{1}{2\pi i} \int_C f(z)\, dz$$

(ii) If $f(z)$ is of the form $f(z) = \frac{\phi(z)}{\psi(z)}$ where $\psi(z_0) = 0$ but $\phi(z_0) \neq 0$, then $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \frac{(z-z_0)\phi(z)}{\psi(z)}$

$$= \lim_{z \to z_0} (z-z_0) \frac{[\phi(z_0) + (z-z_0)\phi'(z_0) + \dots]}{[\psi(z_0) + (z-z_0)\psi'(z_0) + \dots]}$$ By Taylor's theorem

$$= \lim_{z \to z_0} \frac{(z-z_0)\phi(z_0) + (z-z_0)^2 \phi'(z_0) + \dots}{(z-z_0)\psi'(z_0) + \dots} \quad [\because \psi(z_0) = 0]$$

Hence the residue of $f(z) = \frac{\phi(z)}{\psi(z)}$ at $z = z_0$ is $\frac{\phi(z_0)}{\psi'(z_0)}$

• Residue at a pole of order m

If $f(z)$ is analytic within a curve C and has a pole of order m at $z = z_0$, then the residue at $z = z_0$ is

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• Zero of m^{th} order:

If an analytic function $f(z)$ can be expressed in the form $f(z) = (z-a)^m \phi(z)$ where $\phi(z)$ is analytic and $\phi(a) \neq 0$ then $z=a$ is called zero of m^{th} order of the function $f(z)$

A zero of order 1 is called a simple zero.

• Singularity: A singularity of a function $f(z)$ is a point at which the function ceases to be regular (or) analytic.

• Types of Singularities:

• Removable Singularity: If the single valued function is not defined at $z=a$ and $\lim_{z \to a} f(z)$ exists then $z=a$ is a removable singularity (or) if the principle part of Laurant series contains no terms i.e $b_n = 0$ then the point $z=a$ is called removable singularity.

• Essential Singularity: The principle part of laurent series contains infinite numbers of terms of $z-a$, $z=a$ is called essential singularity (or) $z=a$ is an essential singularity if $\lim_{z \to a} f(z)$ does not exist.

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