Analytic Function :-

Let a function $f(z)$ be derivable at every pnt z in an ε neighbourhood of z_0.

i.e., $f'(z)$ exists for all z such that $|z-z_0|<\varepsilon$ where $\varepsilon>0$

Then, $f(z)$ is said to be analytic at z_0.

Problems :-

1. Show that $f(z) = z^3$ is analytic for all z.

Sol :- Let $z = x+iy$

$$\Rightarrow z^3 = (x+iy)^3 = (x^3-3xy^2)+i(3x^2y-y^3)$$

If $z^3 = u+iv$, then we have $u = x^3-3xy^2$, $V = 3x^2y-y^3$.

$$\frac{\partial u}{\partial x} = 3x^2-3y^2 \; ; \; \frac{\partial V}{\partial y} = 3x^2-3y^2$$

$$\frac{\partial u}{\partial y} = -6xy \; , \; \frac{\partial V}{\partial x} = 6xy$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial V}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial V}{\partial x}.$$

Cauchy-Riemann equations are satisfied and the partial derivatives are continuous. Hence $f(z) = z^3$. is analytic.

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2. If $w = \log z$. find $\frac{dw}{dz}$ and determine where w is non-analytic.

Sol :- Let $w = u+iv$

We have $z = x+iy = r.e^{i\theta}$, where $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$

$$\therefore \log(x+iy) = \log(r.e^{i\theta}) = \log r + i\theta = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$$

$$\therefore w = u+iv = \log(x+iy)$$

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$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$ so that

$u = \frac{1}{2}\log(x^2+y^2)$, $v = \tan^{-1}(y/x)$

$\therefore \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2}$, $\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$

$\frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$ and $\frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2}$

The C-R equations are satisfied. Also the partial derivatives are continuous except at (0,0).

$\therefore w = \log z$ is analytic everywhere except at $z=0$.

Hence $\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$= \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2} = \frac{x-iy}{x^2+y^2} = \frac{1}{x+iy} = \frac{1}{z}$ $(z \neq 0)$.

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3. Find all values of K, such that $f(z) = e^x(\cos ky + i\sin ky)$ is analytic.

Sol:- Let $f(z) = u+iv$

$= e^x(\cos ky + i\sin ky)$.

$u(x,y) = e^x\cos ky$, $v(x,y) = e^x\sin ky$.

$\therefore u_x = e^x\cos ky$, $u_y = -ke^x\sin ky$

and $v_x = e^x\sin ky$, $v_y = ke^x\cos ky$

$\Rightarrow \boxed{u_x = v_y \text{ , \& } u_y = -v_x}$

$\left[u_x = \frac{\partial u}{\partial x} \Rightarrow u_y = \frac{\partial u}{\partial y}\right]$

$\left[v_x = \frac{\partial v}{\partial x}, v_y = \frac{\partial v}{\partial y}\right]$

The C-R eqns are satisfied. (i.e),

$\Rightarrow u_x = v_y \Rightarrow e^x\cos ky = ke^x\cos ky$

$\Rightarrow$ which is true for $k = 1$.

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and $U_y = -V_x$

i.e., $-Ke^x \sin ky = -e^x \sin ky$ which is true for $K=1$

Thus $f(z)$ is analytic when $K=1$.

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4. prove that the function $f(z)$ defined by

$$f(z) = \begin{cases} \dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, (z \neq 0) \\ 0, (z=0) \end{cases}$$

is continuous and the C-R equations are satisfied at the Origin, yet $f'(0)$ does not exist.

Sol:- $$\lim_{z\to 0} f(z) = \lim_{\substack{x\to 0 \\ y\to 0}} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$$

$$= \lim_{y\to 0} \frac{-y^3(1-i)}{y^2} = \lim_{y\to 0} [-y(1-i)] = 0$$

and $$\lim_{z\to 0} f(z) = \lim_{\substack{y\to 0 \\ x\to 0}} = \frac{x3(1+1)-y3(1-i)}{x^2+y^2}$$

$$= \lim_{x\to 0} \frac{x^3(1+i)}{x^2} = 0$$

Also $f(0)=0$ by given data.

When $x\to 0, y\to 0$ and $y\to 0, x\to 0$

Now take both x and y tend to zero simultaneously along the path $y=mx$, then

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{x3(1+i)-y3(1-i)}{x^2+y^2}$$

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$$= \lim_{x \to 0} \frac{x^3(1+i) - m^3x^3(1-i)}{(1+m^2)x^2}$$

$$= \lim_{x \to 0} x \frac{[1+i-m^3(1-i)]}{1+m^2} = 0$$

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If $f(z) = u+iv$ be an analytic function of z and if $u-v = (x-y)(x^2+4xy+y^2)$, find $f(z)$ in terms of z.

Sol: Given $f(z) = u+iv$ — (1) $\quad i\cdot f(z) = iu - v$ — (2)

(1)+(2) gives, $f(z)(i+1) = (u-v)+i(u+v)$

$\Rightarrow (1+i)f(z) = U+iV$ where $U = u-v$, $V = u+v$

Differentiating,

$$(1+i)f'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$$

$$= \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} - i\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$$

$$= (x^2+4xy+y^2) + (x-y)(2x+4y) - i\left[-(x^2+4xy+y^2) + (x-y)(4x+2y)\right]$$

$$\therefore (1+i)f'(z) = 3x^2+6xy-3y^2 - i\{3x^2-6xy-3y^2\}$$

Using Milne-Thomson method,

$$(1+i)f'(z) = 3z^2 - i3z^2 \quad (\text{putting } x=z \text{ and } y=0)$$

$$= (1-i)\cdot 3z^2$$

Integrating, $(1+i)f'(z) = 3z^2 - i3z^2$

$$(1+i)f(z) = (1-i)z^3 + c$$

$$\therefore f(z) = \frac{1-i}{1+i}z^3 + \frac{c}{1+i} = \frac{(1-i)^2}{1+1}z^3 + c$$

$$= -iz^3 + c \quad \text{where } c \text{ is a complex constant}$$

In a two-dimensional flow of a fluid, the velocity

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potential is $\phi = x^2 - y^2$. Find the stream function ψ.

Sol:- Since ϕ is velocity potential, it must satisfy the laplace's equation, $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$

Now, $\frac{\partial\phi}{\partial x} = 2x$, $\frac{\partial^2\phi}{\partial x^2} = 2$ and $\frac{\partial\phi}{\partial y} = -2y$, $\frac{\partial^2\phi}{\partial y^2} = -2$.

clearly $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$.

Hence ϕ satisfies the laplace's equation.

Now ψ is the stream function.

Let $W = \phi + i\psi$

$\therefore \frac{dW}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = \frac{\partial\phi}{\partial x} - i\frac{\partial\phi}{\partial y}$ (using CR eqn's)

$2x + i2y = 2(x+iy) = 2z$

Integrating, $W = 2\int z\,dz + c = z^2 + c$ where c is a complex constant.

or $\phi + i\psi = (x+iy)^2 + c = (x^2 - y^2) + i\,2xy + A + iB$

(where $c = A + iB$)

Equating imaginary parts on both sides, we get

$\psi = 2xy + B$ which is the required stream function.

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