UNIT - II

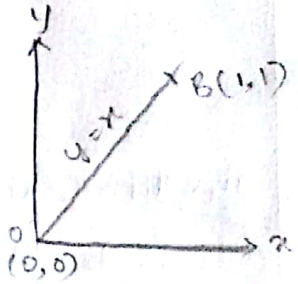
COMPLEX INTEGRATION

1. Evaluate $\int_0^{1+i} (x^2 - iy)\,dz$ along the paths (i) $y = x$ (ii) $y = x^2$.

Sol: i, Along OB whose equation is $y = x \Rightarrow dy = dx$ and x varies from 0 to 1.

$$\int_0^{1+i} (x^2 - iy)\,dz = \int_{(0,0)}^{(1,1)} (x^2 - iy)(dx + idy)$$

$$\therefore \int_{OB} (x^2 - iy)\,dz = \int_{x=0}^{1} (x^2 - iy)(dx + idy) = \int_{x=0}^{1} (x^2 - ix)(dx + idx)$$



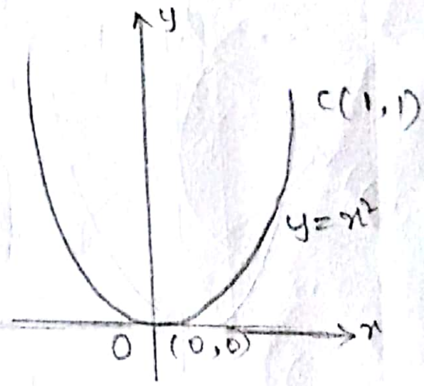
$$= (1+i)\int_0^1 (x^2 - ix)\,dx = (1+i)\left[\frac{x^3}{3} - i\frac{x^2}{2}\right]_0^1$$

$$= (1+i)\left[\frac{1}{3} - \frac{i}{2}\right]$$

ii, Along the parabola whose equation is $y = x^2$

$\therefore dy = 2x\,dx$

Now $$\int_0^{1+i} (x^2 - iy)\,dz = \int_{(0,0)}^{(1,1)} (x^2 - iy)(dx + idy)$$



$$\int_{OC} (x^2 - iy)\,dz = \int_{x=0}^{1} (x^2 - ix^2)(dx + i2x\,dx)$$

$$= (1-i)\int_{x=0}^{1} x^2(1 + 2ix)\,dx$$

$$= (1-i)\int_{x=0}^{1} (x^2 + 2ix^3)\,dx$$

$$= (1-i)\left[\frac{x^3}{3} + i\frac{x^4}{2}\right] = (1-i)\left[\frac{1}{3} + \frac{i}{2}\right]$$

2. (a) Integrate $f(z) = x^2 + ixy$ from $A(1,1)$ to $B(2,8)$ along.

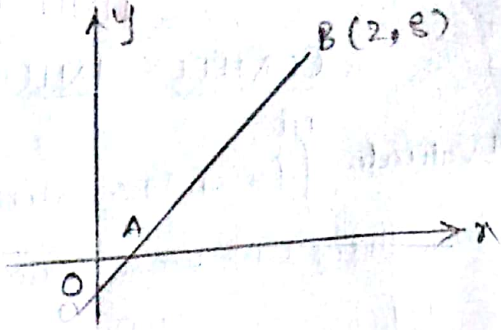
i, The straight line AB ii, The curve $C: x = t,\ y = t^3$

(b) Integrate $f(z) = x^2 + ixy$ from $A(1,1)$ to $B(2,4)$ along curve $x = t,\ y = t^2$

Sol:- i, $$\int f(z)\,dz = \int_{(1,1)}^{(2,8)} (x^2 + ixy)(dx + idy)$$

Along AB: Equation of AB passing through $A(1,1)$ and $B(2,8)$ is

$$\frac{y-1}{8-1} = \frac{x-1}{2-1} \Rightarrow y = 7x-6 \Rightarrow dy = 7dx$$



$$\therefore \int_{AB} f(z)\,dz = \int_{x=1}^{2} \{x^2 + ix(7x-6)\}\{dx + 7i\,dx\}$$

$$= (7i+1)\int_{x=1}^{2} (7i+1)x^2 - 6ix\,dx$$

$$= (7i+1)\left[(7i+1)\frac{x^3}{3} - 3ix^2\right]_1^2 = (7i+1)\left[(7i+1)\frac{8}{3} - 12i - \frac{(7i+1)}{3} + 3i\right]$$

$$= (7i+1)\left[(7i+1)\cdot\frac{7}{3} - 9i\right] = \frac{7i+1}{3}[49i + 7 - 27i] = \frac{7i+1}{3}(22i+7)$$

ii, Along C whose parametric equations are $x = t,\ y = t^3$

$\therefore dx = dt,\ dy = 3t^2 dt$

$A(1,1) \Rightarrow t = 1$ and $B(2,8) \Rightarrow t = 2$

$$\int_C f(z)\,dz = \int_{(1,1)}^{(2,8)} (x^2 + ixy)(dx + i\,dy)$$

$$\therefore \int_C f(z)\,dz = \int_{t=1}^{t=2} (t^2 + it^4)(dt + i3t^2\,dt) = \int_1^2 (t^2 + it^4)(1 + 3it^2)\,dt$$

$$= \int_1^2 (t^2 + it^4 + 3it^4 - 3t^6)\,dt = \int_1^2 [t^2 + (1+3i)t^4 - 3t^6]\,dt$$

$$= \left[\frac{t^3}{3} + (1+3i)\frac{t^5}{5} - \frac{3}{7}t^7\right]_1^2$$

$$= \frac{8}{3} + (1+3i)\frac{32}{5} - \frac{3}{7}(128) - \frac{1}{3} - \frac{(1+3i)}{5} + \frac{3}{7}$$

$$= \frac{1}{105}(-4818 + i1953)$$

Evaluate $\int_C (y^2 + 2xy)\,dx + (x^2 - 2xy)\,dy$ where C is the boundary of the region by $y = x^2$ and $x = y^2$.

Given curves are $y = x^2$ ... ① and $x = y^2 \rightarrow$ ②

The two curves (1) and (2) intersect at points (0, 0) and (1,1)

The positive direction in traversing C is as shown in figure.

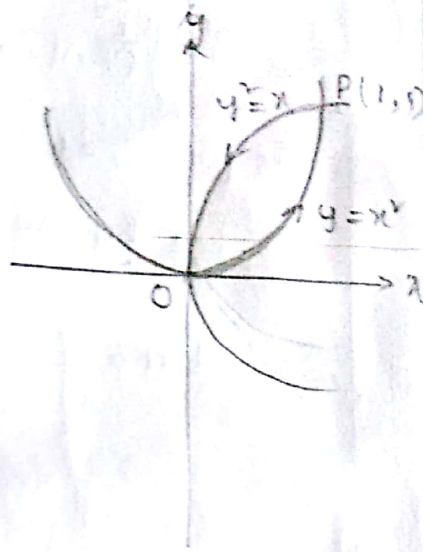
Along $y = x^2$,

the line integral is

$$= \int_{x=0}^{1} (x^4+2x^3)\,dx + (x^2-2x^3)\,d(x^2)$$

$$= \int_0^1 (4x^3-3x^4)\,dx = \left[4\cdot\frac{x^4}{4} - 3\cdot\frac{x^5}{5}\right]_0^1$$

$$= \left[x^4 - \frac{3}{5}x^5\right]_0^1 = 1-\frac{3}{5} = \frac{2}{5} \rightarrow (3)$$



Along $y^2=x$,

the line integral is

$$= \int_{x=1}^{0} (x+2x^{3/2})\,dx + (x^2-2x^{3/2})\,d(\sqrt{x})$$

$$\left\{ (\text{or}) \int_{y=1}^{0} (y^4+2y^3)\,2y\,dy + (y^4-2y^3)\,dy \right\}$$

$$= \int_1^0 (x+2x^{3/2})\,dx + (x^2-2x^{3/2})\cdot\frac{1}{2\sqrt{x}}\,dx$$

$$= \frac{5}{2}\int_1^0 x^{3/2}\,dx = \frac{5}{2}\left[\frac{x^{5/2}}{5/2}\right]_1^0 = 0-1 = -1 \rightarrow (4)$$

Hence the line integral over $C = \frac{2}{5} - 1 = \frac{-3}{5}$ [Adding (3) and (4)]

4\. Evaluate (i) $\int \frac{e^{2z}}{(z-1)(z-2)}\,dz$ where c is the circle $|z|=3$

(ii) $\int \frac{e^{z}}{(z-1)(z-2)}\,dz$ where c is $|z|=2$

Sol. (i) $\frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$ using partial fractions.

$$\therefore \oint_C \frac{e^{2z}\,dz}{(z-2)(z-1)} = \oint_C \frac{e^{2z}}{(z-2)}\,dz - \oint_C \frac{e^{2z}}{(z-1)}\,dz$$

The points $z=1,2$ lies inside c.

Because e^{2z} is analytic everywhere, according to Cauchy's integral formula

$$\oint_C \frac{e^{2z}}{(z-2)}\,dz - \oint \frac{e^{2z}}{(z-1)}\,dz$$

$$= \left[2\pi i e^{2z}\right]_{z=2} - \left[2\pi i e^{2z}\right]_{z=1}$$

$$= 2\pi i\left[e^4 - e^2\right]$$

ii, $\frac{1}{(z-1)(z-4)} = \frac{1}{3}\left[\frac{1}{(z-4)} - \frac{1}{(z-1)}\right]$ using partial fractions

$$\therefore \int_C \frac{e^z}{(z-1)(z-4)}\,dz = \frac{1}{3}\int_C \frac{e^z}{(z-4)}\,dz - \frac{1}{3}\int \frac{e^z}{z-1}\,dz$$

The point $z=4$ lies outside c. But point $z=1$ lies inside c.

$\therefore$ By cauchy's theorem $\int_C \frac{e^z}{z-4}\,dz = 0$

Hence $\int_C \frac{e^z}{(z-1)(z-4)}\,dz = 0 - \frac{1}{3}\int \frac{e^z}{z-1}\,dz$

$= -\frac{1}{3}2\pi i\,(e^z)_{z=1}$, using cauchy's integral theorem

$= -\frac{1}{3}2\pi i e.$

Evaluate $\int_C \frac{z+4}{z^2+2z+5}$ where c is the circle i, $|z|=1$ ii, $|z+1-i|=2$

iii, $|z+1+i|=2$

$$f(z) = \frac{z+4}{(z^2+2z+5)} = \frac{z+4}{(z+1)^2-(2i)^2} = \frac{z+4}{(z+1+2i)(z+1-2i)}$$

Hence the singularities of the function $\frac{z+4}{z^2+2z+5}$ are $-1-2i$ and $-1+2i$. So the function is not analytic at the points $(-1,-2)$ and $(-1,2)$.

i, When c is the circle $|z|=1$ i,e $x^2+y^2-1=0=f(x,y)$

Both $(-1,-2)$ and $(-1,2)$ lie outside the circle because of substituting the values of x and y in the equations of the circle, $f(x,y)>0$ in both the cases.

$\therefore f(z) = \frac{z+4}{z^2+2z+5}$ is analytic at all points within and on the circle $|z|=1$.

$\therefore$ By cauchy's integral theorem, $\int \frac{z+4}{z^2+2z+5}\,dz = 0$

i, When c is the circle $|z+1-i|=2$ i,e $(x+1)^2+(y-1)^2=4$

The point $(-1,-2)$ i,e $z=-1-2i$ does not lie within c whereas the point $(-1,2)$ i,e $z=-1+2i$ lies inside.

$$\therefore \int_C \frac{z+4}{(z+1+2i)(z+1-2i)}\,dz = \int \frac{\frac{z+4}{z+1+2i}}{z+1-2i}\,dz$$

$$= \int \frac{f(z)}{(z-a)}\,dz \text{ where } f(z) = \frac{z+4}{z+1+2i} \text{ and } a = -1+2i$$

By Cauchy's integral formula $\int \frac{f(z)}{(z-a)}\,dz = 2\pi i f(a) \rightarrow (1)$

But $f(z) = \frac{z+4}{(z+1+2i)}$, $a = -1+2i$

$\therefore f(a) = f(-1+2i) = \frac{-1+2i+4}{-1+2i+1+2i} = \frac{3+2i}{4i}$

Substituting in equ (1), we get

$$\int_C \frac{z+4}{z^2+2z+5}\,dz = 2\pi i\left[\frac{3+2i}{4i}\right] = \frac{\pi}{2}(3+2i)$$

ii, When C is the circle $|z+1+i| = 2$ i.e $(x+1)^2+(y+1)^2 = 4$

The point $(-1,-2)$ i.e $z = -1-2i$ lies inside whereas the point $z = -1+2i$ lies out-side the circle.

$$\therefore \int \frac{z+4}{(z+1+2i)(z+1-2i)}\,dz = \int \frac{\frac{z+4}{z+1-2i}}{z+1+2i}\,dz = \int_C \frac{f(z)}{(z-a)}\,dz$$

where $f(z) = \frac{z+4}{z+1-2i}$ and $a = -1-2i$

Hence $\int_C \frac{z+4}{(z+1+2i)(z+1-2i)}\,dz = 2\pi i f(a) = 2\pi i f(-1-2i)$

$$= 2\pi i\left[\frac{-1-2i+4}{-1-2i-2i+4}\right] = 2\pi i\left[\frac{3-2i}{-4i}\right] = \frac{\pi}{2}(2i-3)$$

~~Evaluate~~ $\int \frac{z^2-z-1}{z(z-1)^2}$

Using Cauchy's integral formula, evaluate $\int \frac{z}{(z-1)(z-2)^2}\,dz$, where $C: |z-2| = 1/2$

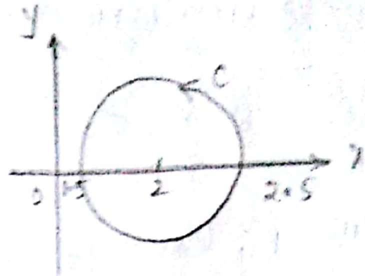
The integral has two singular points at $z=1$ and $z=2$ of which only $z=2$ lies inside C.

$f(z) = \frac{z}{(z-1)}$ is analytic on and which C.

Here $a = 2$ and $n=1$

$\therefore$ By Cauchy's integral formula

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz,$$



we get

$$\int_C \frac{z}{(z-1)(z-2)^2}\,dz = 2\pi i\left[\frac{d}{dz}\left[\frac{z}{z-1}\right]\right]_{z=2}$$

$$= 2\pi i\left[\frac{d}{dz}\left[1+\frac{1}{z-1}\right]\right]_{z=2}$$

$$= 2\pi i\left[\frac{-1}{(z-1)^2}\right]_{z=2} = -2\pi i$$

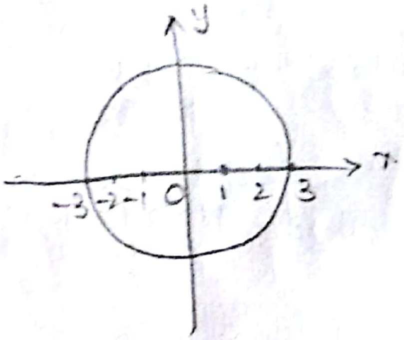
7. Use Cauchy's integral formula to evaluate $\oint_C \frac{e^z}{(z+2)(z+1)^2}\,dz$ where C is the circle $|z|=3$

Sol: Singular points of integrands are given by putting the denominator equal to zero.

$(z+2)(z+1)^2 = 0 \Rightarrow z = -1, -2$

Both singular points are inside the given circle with centre at origin and radius 3

$$\therefore \int \frac{e^z}{(z+2)(z+1)^2}\,dz = \int \frac{e^z}{(z+2)(z+1)^2}\,dz + \int \frac{e^z}{(z+2)(z+1)^2}\,dz$$



$$= \int \frac{\frac{e^z}{z+2}}{(z+1)^2}\,dz + \int \frac{\frac{e^z}{(z+1)^2}}{z+2}\,dz$$

$$= 2\pi i f'(-1) + 2\pi i f(-2),$$

by Cauchy's Integral formula

$$= 0 + 2\pi i \frac{e^{-2}}{(-2+1)^2} = \frac{2\pi i}{e^2}$$

8. <u>Taylors Theorem Statement:-</u>

If a function $f(z)$ is analytic inside a circle 'c' whose centre is 'a' then for all values of z inside C $f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a)$

$$+ \cdots + \frac{(z-a)^n}{n!}f^n(a) + \cdots$$

Laurent Series Statement:- If $f(z)$ is analytic inside and on the boundary of ring shaped region R bounded by 2 concentric circles C_1 & C_2 of radius r_1 and r_2 respectively $(r_1 > r_2)$ having centre at a then for all z in 'R' $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_1(z-a)^{-1} + a_{-2}(z-a)^{-2} + \cdots$

$$a_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(w)}{(w-a)^{n+1}}\,dw, \quad n = 0,1,2$$

$$a_{-n} = \frac{1}{2\pi i}\oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}}\,dw, \quad n = 1,2,3$$

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{Analytic part}} + \underbrace{\sum_{n=1}^{\infty} a_{-n}(z-a)^{-n}}_{\text{principle part.}}$$

9. Find Taylor's series expansion for the function $f(z) = \frac{1}{(1+z)^2}$ with centre at $-i$.

Sol:- By Taylors theorem

$$f(z) = f(a) + f(z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \cdots \frac{(z-a)^n}{n!}f^n(a) + \cdots$$

Put $a = -i$

$$\therefore f(z) = f(-i) + f(z+i)f'(-i) + \frac{(z+i)^2}{2!}f''(-i) + \cdots + \frac{(z+i)^n}{n!}f^n(-i) + \cdots \rightarrow (1)$$

Here $f(z) = \frac{1}{(1+z)^2}$

$$\therefore f^n(z) = \frac{(-1)^n (n+1)!}{(1+z)^{n+2}}$$

Now $f(-i) = \frac{1}{(1-i)^2} = \frac{i}{2}$ and $f^n(-i) = \frac{(-1)^n (n+1)!}{(1-i)^{n+2}}$

Substituting in equ(1), we get

$$\frac{1}{(1+z)^2} = \frac{i}{2} + \sum_{n=1}^{\infty} \frac{(z+i)^n}{n!}\frac{(-1)^n (n+1)!}{(1-i)^{n+2}} = \frac{i}{2} + \sum_{n=1}^{\infty} (-1)^n (n+1)\frac{(z+i)^n}{(1-i)^{n+2}}$$

$$= \frac{i}{2} + \sum_{n=1}^{\infty} (-1)^n (n+1)\frac{(z+i)^n}{(1-i)^n}\frac{1}{(1-i)^2} = \frac{i}{2} + \sum_{n=1}^{\infty} (-1)^n (n+1)\frac{(z+i)^n}{(1-i)^n(-2i)}$$

$$= \frac{i}{2} + \frac{i}{2}\sum_{n=1}^{\infty} (-1)^n (n+1)\frac{(z+i)^n}{(1-i)^n}$$

Obtain the Taylors Series to represent the function $\frac{z^2-1}{(z+2)(z+3)}$

Let $f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3}$ Resolving into partial fractions

$$= 1 + \frac{3}{2(1+\frac{z}{2})} - \frac{8}{3(1+\frac{z}{3})} = 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

Expanding by Binomial Series,

$$f(z) = 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right]$$

$$= 1 + \frac{3}{2}\sum_{z=0}^{\infty} \frac{(-1)^n}{2^n} z^n - \frac{8}{3}\sum_{z=0}^{\infty} (-1)^n \frac{z^n}{3^n}$$

$$= 1 + \sum_{z=0}^{n} (-1)^n \left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right] z^n$$

Expand $f(z) = \frac{1}{z^2-3z+2}$ in the region (i) $0<|z-1|<1$ (ii) $1<|z|<2$

: We have

$$f(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)}$$

$$= \frac{(z-1)-(z-2)}{(z-1)(z-2)} = \frac{1}{z-1} - \frac{1}{z-2}$$

∴ $z=1$, $z=2$ are the singular points of $f(z)$.

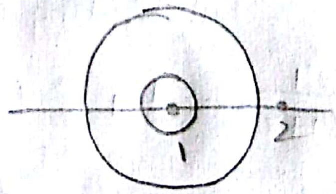
(i) The function $f(z)$ is analytic in the ring shaped region $0<r_2<|z-1|<r_1$

Put $z-1=w$ ∴ $z=w+1 \Rightarrow z-2 = w+1-2 = w-1$

$$\therefore f(z) = \frac{1}{w-1} - \frac{1}{w} = -\frac{1}{w} - \frac{1}{1-w}$$

$$= -\frac{1}{w} - [1+w+w^2+\cdots] \text{ if } w<1 \text{ and } w\neq 0$$

$$= -\frac{1}{(z-1)} - \sum_{n=0}^{\infty}(z-1)^n \text{ if } 0<|z-1|<1$$



$$= -\sum_{n=-1}^{\infty}(z-1)^n \text{ if } 0<|z-1|<1$$

(ii) Given $1<|z|<2$ i.e $1<|z|$ and $|z|<2$ or $\left|\frac{1}{z}\right|<1$ and $\left|\frac{z}{2}\right|<1$

$$\therefore f(z) = \frac{1}{z-2} - \frac{1}{z-1} = \frac{1}{-2\left[1-\frac{z}{2}\right]} - \frac{1}{z\left[1-\frac{1}{z}\right]} = -\frac{1}{2}\left[1-\frac{z}{2}\right]^{-1} - \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right] - \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]$$

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right] - \left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]$$

Expand $\frac{7z-2}{(z+1)z(z-2)}$ about the point $z=-1$ in the region $1<|z+1|<3$ by Laurent's series.

Let $f(z) = \frac{7z-2}{(z+1)(z)(z-2)}$

put $z+1=w$, then $z=w-1$

$$\therefore f(z) = \frac{7(w-1)-2}{w(w-1)(w-1-2)} = \frac{7w-9}{w(w-1)(w-3)}$$

By partial fractions,

$$f(z) = \frac{A}{w}+\frac{B}{w-1}+\frac{C}{w-3} = \frac{3}{w}+\frac{1}{w-1}+\frac{2}{w-3}$$

$$= \frac{-3}{w}+\frac{1}{w\left[1-\frac{1}{w}\right]}-\frac{2}{3\left[1-\frac{w}{3}\right]} = \frac{3}{w}+\frac{1}{w}\left[1-\frac{1}{w}\right]^{-1}-\frac{2}{3}\left[1-\frac{w}{3}\right]^{-1}$$

$$= -\frac{3}{w}+\frac{1}{w}\left[1+\frac{1}{w}+\frac{1}{w^2}+\cdots\right]-\frac{2}{3}\left[1+\frac{w}{3}+\frac{w^2}{3^2}+\cdots\right]$$

$$= \left[-\frac{2}{w}+\frac{1}{w^2}+\frac{1}{w^3}+\cdots\right]-\frac{2}{3}\left[1+\frac{w}{3}+\frac{w^2}{3^2}+\cdots\right]$$

$$= \frac{-2}{(z+1)}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}\cdots-\frac{2}{3}\left[1+\frac{(z+1)}{3}+\frac{(z+1)^2}{3^2}+\cdots\right]$$

The above series is valid in the region $\left|\frac{1}{w}\right|<1$ and $\left|\frac{w}{3}\right|<1$

i,e $1<|w|$ and $|w|<3$ i,e $1<|w|<3$ or $1<|z+1|<3$.

1. Find the poles and residues of $\frac{1}{z^2-1}$. 6

Sol: Let $f(z) = \frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$, using partial fractions. poles of $f(z)$ are given by $(z-1)(z+1)=0$ i.e $z = \pm 1$. These are simple poles. To find the residue at $z=1$, we expand the function in a laurent series in powers of $z-1$.

To expand in powers of $z-1$, we write

$$f(z) = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z-1+2}\right) = \frac{1}{2}\left(\frac{1}{u} - \frac{1}{u+2}\right) \text{ where } u = z-1$$

$$= \frac{1}{2u} - \frac{1}{4}\left(\frac{1}{1+\frac{u}{2}}\right) = \frac{1}{2u} - \frac{1}{4}\left(1+\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{2u} - \frac{1}{4}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right)$$

$$= \frac{1}{2u} - \frac{1}{4} + \frac{u}{8} - \frac{u^2}{16} + \frac{u^3}{32} - \cdots$$

$$= \frac{-1}{4} + \frac{1}{2(z-1)} + \frac{z-1}{8} - \frac{(z-1)^2}{16} + \cdots$$

coefficient of $\frac{1}{z-1} = \frac{1}{2}$ and hence residue at $z=1$ is $\frac{1}{2}$

To find the residue at $z=-1$, we expand $f(z)$ in a laurent series in powers of $z+1$.

we have $f(z) = \frac{1}{2}\left(\frac{1}{z+1-2} - \frac{1}{z+1}\right) = \frac{1}{2}\left(\frac{1}{u-2} - \frac{1}{u}\right)$ where $u = z+1$

$$= \frac{-1}{4}\left(\frac{1}{1-\frac{u}{2}}\right) - \frac{1}{2u} = \frac{-1}{4}\left(1-\frac{u}{2}\right)^{-1} - \frac{1}{2u}$$

$$= \frac{-1}{4}\left(1 + \frac{u}{2} + \frac{u^2}{4} + \frac{u^3}{8} + \cdots\right) - \frac{1}{2u}$$

$$= -\frac{1}{4}\left[1 + \frac{x+1}{2} + \frac{(x+1)^2}{4} + \frac{(x+1)^3}{8} + \dots\right] - \frac{1}{2(x+1)}$$

Thus coefficient of $\frac{1}{x+1} = \frac{-1}{2}$ $\therefore$ Residue at $x=1$ is $-\frac{1}{2}$.

2.) Find the residue of $\frac{1}{(x-\sin x)}$ at $x=0$.

Solution: Let $f(x) = \frac{1}{x-\sin x}$

The function $f(x)$ has a laurent's expansion.

$$f(x) = \frac{1}{x - \left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right]} = \frac{1}{\frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \dots}$$

$$= \frac{1}{\frac{x^3}{3!}\left(1 - \frac{x^2}{20} + \frac{x^4}{840} - \dots\right)} = \frac{6}{x^3}\left[1 - \left(\frac{x^2}{20} - \frac{x^4}{840} + \dots\right)^{-1}\right]$$

$$= \frac{6}{x^3}\left[1 + \left(\frac{x^2}{20} - \frac{x^4}{840}\right) + \left(\frac{x^2}{20} - \frac{x^4}{840}\right)^2 + \dots\right] = \frac{6}{x^3}\left[1 + \frac{x^2}{20} - \frac{x^4}{840} + \frac{x^4}{400} - \dots\right]$$

$$= \frac{6}{x^3}\left[1 + \frac{x^2}{20} + \frac{20x^4}{840} + \dots\right]$$

$$= \frac{6}{x^3} + \frac{6}{20}\cdot\frac{1}{x} + \frac{120x}{840} + \dots$$

$\therefore$ Residue at the pole $x=0$ is coefficient of $\frac{1}{x} = \frac{6}{20} = \frac{3}{10}$

3.) Determine the poles of the function

(i) $f(x) = \frac{x^2}{(x-1)^2(x+2)}$

(ii) $f(x) = \frac{x^2}{(x+1)^2(x+2)}$ and the residues at each pole.

solution: $x=1$ and $x=-2$ are the zeroes of denominator of order 2 and 1 respectively. $\therefore$ $x=1$ is a pole of order 2 and $x=-2$ is a pole of order 1 of $f(x)$.

(i) $$[\operatorname{Res} f(x)]_{x=-2} = \lim_{x\to -2} (x+2)\, f(x) = \lim_{x\to -2} (x+2) \frac{x^2}{(x-1)^2(x+2)}$$

$$\lim_{x\to -2} \frac{x^2}{(x-1)^2} = \frac{4}{9}$$

$$[\operatorname{Res} f(x)]_{x=1} = \lim_{x\to 1} \frac{1}{1!} \frac{d}{dx} [(x-1)^2 f(x)]$$

$$= \lim_{x\to 1} \frac{d}{dx}\left[(x-1)^2 \cdot \frac{x^2}{(x-1)^2(x+2)}\right] = \lim_{x\to 1}\left[\frac{d}{dx}\left(\frac{x^2}{(x+2)}\right)\right]$$

$$= \lim_{x\to 1}\left[\frac{(x+2)\cdot 2x - x^2 \cdot 1}{(x+2)^2}\right] = \lim_{x\to 1}\left[\frac{x^2+4x}{(x+2)^2}\right] = \frac{5}{9}$$

(ii) $x=-1$ is a pole of order 2 and $x=-2$ is a pole of order one

From (i), it is obvious that $[\operatorname{Res} f(x)]_{x=-2} = \lim_{x\to -2} \frac{x^2}{(x+1)^2} = 4$

and $$[\operatorname{Res} f(x)]_{x=-1} = \lim_{x\to -1}\left[\frac{x^2+4x}{(x+2)^2}\right] = -3$$

4. Find the residue of $\frac{xe^x}{(x-1)^3}$ at its pole.

Solution: Let $f(x) = \frac{xe^x}{(x-1)^3}$

Poles of $f(x)$ are obtained by putting the denominator equal to zero.

$x=1$ is a pole of $f(x)$ of order 3.

We know that if $f(z)$ has a pole of order m at $z=a$, then

$$[\text{Res } f(z)]_{z=a} = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} [(z-a)^m f(z)]$$

Here $a=1, m=3$

$$\therefore [\text{Res } f(z)]_{z=1} = \frac{1}{2!} \lim_{z \to 1} \frac{d^2}{dz^2} [(z-1)^3 f(z)]$$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d^2}{dz^2} (z e^z) = \frac{1}{2} \lim_{z \to 1} \frac{d}{dz} (z e^z + e^z)$$

$$= \frac{1}{2} \lim_{z \to 1} (z e^z + e^z + e^z) = \frac{1}{2} \lim_{z \to 1} e^z (z+2) = \frac{1}{2} e(3)$$

$$= \frac{3e}{2}$$

- Calculation of Residues

(i) when $z = z_0$ is a simple pole.

In this case the laurent's series expansion becomes

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{a_{-1}}{z-z_0}$$

$$\therefore \lim_{z \to z_0} (z-z_0) f(z) = a_{-1}$$

$$\therefore \text{Res}(f : z = z_0) \text{ or } [\text{Res} f(z)]_{z=z_0} = \lim_{z\to z_0} (z-z_0) f(z) = a_{-1}$$

$$= \frac{1}{2\pi i} \int_c f(z)\, dz$$

(ii) If $f(z)$ is of the form $f(z) = \frac{\phi(z)}{\psi(z)}$ where $\psi(z_0)=0$ but $\phi(z_0) \neq 0$, then $\lim_{z\to z_0} (z-z_0) f(z) = \lim_{z\to z_0} \frac{(z-z_0)\phi(z)}{\psi(z)}$

$$= \lim_{z\to z_0} (z-z_0) \frac{[\phi(z_0) + (z-z_0)\phi'(z_0) + \dots]}{[\psi(z_0) + (z-z_0)\psi'(z_0) + \dots]}$$ By Taylor's theorem

$$= \lim_{z\to z_0} \frac{(z-z_0)\phi(z_0) + (z-z_0)^2 \phi'(z_0) + \dots}{(z-z_0)\psi'(z_0) + \dots} \quad [\because \psi(z_0) = 0]$$

Hence the residue of $f(z) = \frac{\phi(z)}{\psi(z)}$ at $z = z_0$ is $\frac{\phi(z_0)}{\psi'(z_0)}$

- Residue at a pole of order m

If $f(z)$ is analytic within a curve C and has a pole of order m at $z = z_0$, then the residue at $z = z_0$ is

$$\lim_{z\to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right]$$

Proof: Given that $f(z)$ has a pole of order m. Therefore $f(z)$ is expressible as $(z-z_0)^m f(z) = \phi(z)$ where $\phi(z)$ is analytic and $\phi(z_0) \neq 0$.

$$\therefore f(z) = \frac{\phi(z)}{(z-z_0)^m} \qquad (1)$$

Residue of $f(z)$ at $z = z_0$ is a_{-1} where

$$a_{-1} = \frac{1}{2\pi i} \int_c f(z)\, dz = \frac{1}{2\pi i} \int_c \frac{\phi(z)}{(z-z_0)^m}\, dz$$

$$= \frac{1}{(m-1)!} (m-1)! \frac{1}{2\pi i} \int_c \frac{\phi(z)}{(z-z_0)^{m-1+1}}\, dz$$

$$= \frac{1}{(m-1)!} \phi^{m-1}(z_0) \quad \left[\text{since } f^n(z_0) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right]$$

$$\therefore a_{-1} = \left[\operatorname{Res} f(z) \right]_{z=z_0} = \lim_{z\to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] \text{ by } (1)$$

- **Definition:**

Let $z = z_0$ be a singularity of $f(z)$ and if there is no other singularity within a small circle surrounding the point $z = z_0$ Then, this point is said to be an isolated singularity and otherwise it is termed as non-isolated singularity

- **Singularities of an Analytic function:**

A zero of an analytic function:

A zero of an analytic function $f(z)$ is a value of z such that $f(z) = 0$.

• Zero of m^{th} order:

If an analytic function $f(z)$ can be expressed in the form $f(z) = (z-a)^m \phi(z)$ where $\phi(z)$ is analytic and $\phi(a) \neq 0$ then $z=a$ is called zero of m^{th} order of the function $f(z)$

A zero of order 1 is called a simple zero.

• Singularity: A singularity of a function $f(z)$ is a point at which the function ceases to be regular (or) analytic.

• Types of Singularities:

• Removable Singularity: If the single valued function is not defined at $z=a$ and $\lim_{z\to a} f(z)$ exists then $z=a$ is a removable singularity (or) of the principle part of Laurent series contains no terms i.e $b_n = 0$ then the point $z=a$ is called removable singularity.

• Essential Singularity: The principle part of laurent series contains infinite numbers of terms of $z-a$, $z=a$ is called essential singularity (or) $z=a$ is an essential singularity if $\lim_{z\to a} f(z)$ does not exist.