12/7/17

Formula

1. $\frac{d}{dx} x^n = n x^{n-1}$

2. $\frac{d}{dx} \frac{1}{x^n} = x^{-n}/n+1$

3. $\frac{d}{dx} (ax+b) = a$

4. $\frac{d}{dx} \log x = \frac{1}{x}$

5. $\frac{d}{dx} e^x = e^x$

6. $\frac{d}{dx} (K) = 0$

7. $\frac{d}{dx} (Kx) = K$

8. $\frac{d}{dx} (Kx^2) = 2Kx$

9. $\frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$

10. $\frac{d}{dx} \left(\frac{1}{x}\right) = \frac{-1}{x^2}$

11. $\frac{d}{dx} (ax+b)^2 = 2(ax+b)(a)$

12. $\frac{d}{dx} (\sqrt{ax+b}) = \frac{a}{2\sqrt{ax+b}}$

13. $\frac{d}{dx} (\log a^x) = \frac{1}{x} \log a$

14. $\frac{d}{dx} \log f(x) = \frac{1}{f(x)} f'(x)$

15. $\frac{d}{dx} \operatorname{cosec}^{-1} x = \frac{-1}{x\sqrt{x^2-1}}$

16) $\frac{d}{dx} e^{f(x)} = e^{f(x)} f'(x)$

17) $\frac{d}{dx} a^x = a^x \log a$

18) $\frac{d}{dx} x^x = x^x (1+\log x)$

19) $\frac{d}{dx} Kx^3 = 3Kx^2$

20) $\frac{d}{dx} (\sin x) = \cos x$

21) $\frac{d}{dx} (\cos x) = -\sin x$

22) $\frac{d}{dx} (\tan x) = \sec^2 x$

23) $\frac{d}{dx} (\cot x) = -\operatorname{cosec}^2 x$

24) $\frac{d}{dx} (\sec x) = \sec x \tan x$

25) $\frac{d}{dx} (\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

26) $\frac{d}{dx} (\sinh x) = \cosh x$

27) $\frac{d}{dx} (\cosh x) = \sinh x$

28) $\frac{d}{dx} (\tanh x) = \operatorname{sech}^2 x$

29) $\frac{d}{dx} (\coth x) = -\operatorname{cosech}^2 x$

30) $\frac{d}{dx} (\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

Integration formulae:

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$
2. $\int x\,dx = \frac{x^2}{2} + c$
3. $\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + c$
4. $\int \frac{1}{x}\,dx = \log|x| + c$
5. $\int e^x\,dx = e^x + c$
6. $\int a^x\,dx = \frac{a^x}{\log a} + c$
7. $\int \cos x\,dx = \sin x + c$
8. $\int \sin x\,dx = -\cos x + c$
9. $\int \sec^2 x\,dx = \tan x + c$
10. $\int \mathrm{cosec}^2 x\,dx = -\cot x + c$
11. $\int \sec x \tan x\,dx = \sec x + c$
12. $\int \mathrm{cosec}\,x \cot x\,dx = -\mathrm{cosec}\,x + c$
13. $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c = -\cos^{-1}x + c$
14. $\int \frac{1}{1+x^2}\,dx = \mathrm{Tan}^{-1}x + c = -\cot^{-1}x + c$
15. $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + c = -\mathrm{cosec}^{-1}x + c \quad (x>1)$
16. $\int \frac{1}{x\sqrt{x^2-1}}\,dx = -\sec^{-1}x + c = \mathrm{cosec}^{-1}x + c \quad (x<-1)$
17. $\int \sinh x\,dx = \cosh x + c$
18. $\int \cosh x\,dx = \sinh x + c$
19. $\int \mathrm{sech}^2 x\,dx = \tanh x + c$
20. $\int \mathrm{cosech}^2 x\,dx = -\coth x + c$
21. $\int \mathrm{sech}\,x \tanh x\,dx = -\mathrm{sech}\,x + c$
22. $\int \mathrm{cosech}\,x \coth x\,dx = -\mathrm{cosech}\,x + c$
23. $\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1}x + c$
24. $\int \frac{1}{\sqrt{x^2-1}}\,dx = \cosh^{-1}x + c$
25. $\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)| + c$
26. $\int \tan x\,dx = \log|\sec x| + c$
27. $\int \cot x\,dx = \log|\sin x| + c$
28. $\int \sec x\,dx = \log|\sec x + \tan x| + c$

    $= \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$
29. $\int \mathrm{cosec}\,x\,dx = \log|\mathrm{cosec}\,x - \cot x| + c$

    $= \log\left|\tan\frac{x}{2}\right| + c$
30. $\int (f(x))^n f'(x)\,dx = \frac{(f(x))^{n+1}}{n+1} + c$

31. $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c$

32. $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

33. $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + c$

34. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right) + c$

35. $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

36. $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c$

37. $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$

38. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

39. $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + c$

40. $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right) + c$

41. $\int uv = u\int v\,dx - \int u'\int v\,dx.$

42. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c$

43. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

44. $\int e^{x}(f(x)+f'(x))\,dx = e^{x}f(x) + c$

45. $\int e^{-x}[f(x)+f'(x)]\,dx = -e^{-x}f(x) + c.$

46. $\int \log x\,dx = x(\log x - 1)$

Trignometric formulae:

1. $\cos(A+B) = \cos A \cos B - \sin A \sin B$

2. $\cos(A-B) = \cos A \cos B + \sin A \sin B$

3. $\sin(A+B) = \sin A \cos B + \cos A \sin B$

4. $\sin(A-B) = \sin A \cos B - \cos A \sin B$

5. $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

6. $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

7. $\cot(A+B) = \dfrac{\cot A \cot B - 1}{\cot B + \cot A}$

8. $\cot(A-B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$

9. $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A.$

10. $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A.$

11. $\tan\left(\frac{\pi}{4}+A\right) = \dfrac{1+\tan A}{1-\tan A} = \dfrac{\cos A + \sin A}{\cos A - \sin A}.$

12. $\tan\left(\frac{\pi}{4}-A\right) = \dfrac{1-\tan A}{1+\tan A} = \dfrac{\cos A - \sin A}{\cos A + \sin A}$

13. $\sin 2A = 2\sin A \cos A = \dfrac{2\tan A}{1+\tan^2 A}$

14. $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \cancel{1-\tan} \dfrac{1-\tan^2 A}{1+\tan^2 A}$

15. $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

16. $1-\cos 2A = 2\sin^2 A$

17. $1+\cos 2A = 2\cos^2 A$.

18. $\sin 3A = 3\sin A - 4\sin^3 A$

19. $\cos 3A = 4\cos^3 A - 3\cos A$

20. $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1-3\tan^2 A}$

21. $\sin(A+B) + \sin(A-B) = 2\sin A \cos B$

22. $\sin(A+B) - \sin(A-B) = 2\cos A \sin B$

23. $\cos(A+B) + \cos(A-B) = 2\cos A \cancel{\sin} \cos B$

24. $\cos(A-B) - \cos(A+B) = 2\sin A \sin B$

UNIT - 1: Functions of Complex Variables

Complex Number:

A complex number 'z' is an ordered pair (x, y) of real numbers and is written as $z = x+iy$ where $i^2 = -1$, $i = \sqrt{-1}$.

1. 'x' is called real part of 'z' & 'y' is called imaginary part of 'z'.

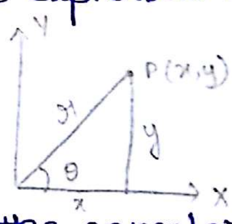
2. In the Argand's diagram, the complex number 'z' is represented by the point $P(x,y)$.

3. If $P(r, \theta)$ are polar co-ordinates of P then $r = \sqrt{x^2+y^2}$ is called modulus of 'z' and is denoted by $|z|$.

4. $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is called the argument of 'z' and is denoted by $\arg(z)$.

5. Every complex number z (non-zero) can be expressed as

$z = r(\cos\theta + i\sin\theta) = r\,\text{cis}\,\theta = re^{i\theta}$



6. If $z = x+iy$ then $\bar{z} = x-iy$ is called the complex conjugate of z, also $|z| = |\bar{z}|$, $|z^2| = z \times \bar{z}$.

Real part of $z = \frac{z+\bar{z}}{2}$, imaginary part of $z = \frac{z-\bar{z}}{2i}$

7. To represent $w = f(z)$ graphically we take two argand diagrams, one ~~two~~ represent the point 'z' and the other to represent 'w' called the z-plane & w-plane.

Function of a Complex Variable:

If for each complex variable $z = x+iy$ in a given region 'R' we have one (or) more values of $w = u+iv$ then 'w' is said to be a function of 'z' and we write $w = u(x,y) + iv(x,y) = f(z)$ where u, v are functions of x & y.

Ex: If $w = z^2$ where $z = x+iy$ and $w = f(z) = u+iv$.

$w = u+iv = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$

$\therefore u = x^2 - y^2$ and $v = 2xy$.

u and v are real and imaginary parts of 'w', are functions of real variables 'x' & 'y'.

Note:

1. If to each value of 'z' there corresponds one and only value of 'w' then w is called a single valued function of 'z'.

2. If more than one value then 'w' is called multivalued function of 'z'.

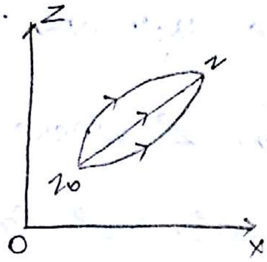
Limit of f(z):

A function $f(z)$ tends to the limit 'l' as $z \to z_0$ along any path, if to each positive arbitary number 'ϵ' there corresponds a positive number 'δ' such that $|f(z) - l| < \epsilon$ whenever $0 < |z - z_0| < \delta$ i.e. $l - \epsilon < f(z) < l + \epsilon$ whenever $z_0 - \delta < z < z_0 + \delta$, $z \neq z_0$ then $\underset{z \to z_0}{Lt} f(z) = l$

Note :

In real variables $x \to x_0 \Rightarrow x$ approaches x_0 along the number line either from left (or) right.

In complex variables, $z \to z_0 \Rightarrow z$ approaches z_0 along any path straight (or) curved.



Continuity of f(z):

A single valued function $f(z)$ is said to be continous at a point $z = z_0$ if $\underset{z \to z_0}{Lt} f(z) = f(z_0)$.

function $f(z)$ is said to be continous in a region 'R' of the 'z' plane if it is continous at every point of the region.

Derivative of f(z):

Let $w = f(z)$ be a single valued function of the variable $z = x + iy$ then the derivative of $w = f(z)$ is defined as

$$\frac{dw}{dz} = f'(z) = \underset{\delta z \to 0}{LT} \frac{f(z+\delta z) - f(z)}{\delta z}$$, provided the limit exists & has the same value for all the different ways in which δz approaches zero. $(\delta z \to 0)$.

Note :

If $f(z)$ is differenciable at z_0 then $f(z)$ is continous at $z = z_0$, but the converse of the other theorem is not true for all the cases.

Analytic functions:

If a single valued function $f(z)$ posses a unique derivative at every point in the region 'R' then $f(z)$ is said to be an analytic (or) holomorphic function (or) regular fun^c in R.

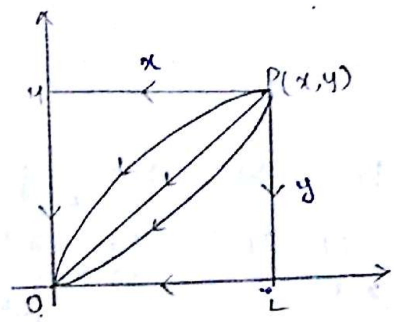
A point where the fun^c does not posses a derivative is called a singular point of the function.

A function which is analytic everywhere is known as an entire function.

Ex: Polynomials are entire functions since the derivative exist in the entire complex plane.

Q. Show that $f(x) = \begin{cases} \dfrac{x^3-y^3}{x^3+y^3} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ is discontinous at origin.

Sol: Given, $f(x) = \begin{cases} \dfrac{x^3-y^3}{x^3+y^3}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$



Let $z \to 0$ along PLO.

Along PL, y varies

Along LO, x varies

Along PLO:

$\underset{z\to 0}{Lt}\, f(z) = \underset{\substack{y\to 0\\ x\to 0}}{Lt} \dfrac{x^3-y^3}{x^3+y^3} = \underset{x\to 0}{Lt} \dfrac{x^3-0}{x^3+0} = \underset{x\to 0}{Lt} \dfrac{x^3}{x^3} = \underset{x\to 0}{Lt}\, 1 = 1$ —— (1)

let $z \to 0$, along PMO,

Along, PM, x varies

MO, y varies

$$\underset{z\to 0}{Lt}\, f(z) = \underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{x^3-y^3}{x^3+y^3} = \underset{y\to 0}{Lt} \frac{-y^3}{y^3} = -1 \quad \text{—— (2)}$$

$\therefore$ from (1) & (2), we can say that limit is not unique.

$\therefore f(z)$ is discontinous at zero.

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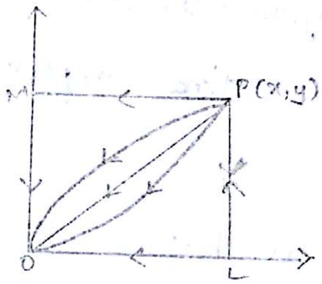
2) $$f(z) = \frac{x^2 y(y-x)}{(x^6+y^2)(x+y)}, \text{ if } (x,y) \neq (0,0)$$

$$= 0 \quad , \text{ if } (x,y) = (0,0)$$



Sol: Let $z \to 0$, along PLO,

$$\underset{z\to 0}{Lt}\, f(z) = \underset{\substack{y\to 0\\ x\to 0}}{Lt} \frac{x^2y(y-x)}{(x^6+y^2)(x+y)} = 0$$

Let $z \to 0$, along PMO.

$$\underset{z\to 0}{Lt}\, f(z) = \underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{x^2y(y-x)}{(x^6+y^2)(x+y)} = 0.$$

Let $z \to 0$, along $y = mx$ (straight lines).

$$\underset{z\to 0}{Lt}\, f(z) = \underset{x\to 0}{Lt} \frac{x^2(mx).(mx-x)}{[x^6+(mx)^2][x+mx]}$$

$$= \underset{x\to 0}{Lt} \frac{x^4 m(m-1)}{x^3(x^4+m^2)(1+m)}$$

$$= \underset{x\to 0}{Lt} \frac{xm(m-1)}{(x^4+m^2)(1+m)} = 0.$$

Let $z \to 0$ along $y = mx^2$,

$$\underset{x\to 0}{Lt} f(z) = \underset{x\to 0}{Lt} \frac{x^2(mx^2)(mx^2-x)}{(x^6+(mx^2)^2)[x+mx^2]}$$

$$= \underset{x\to 0}{Lt} \frac{\cancel{x^5}.m(mx-1)}{\cancel{x^5}(x^2+m^2)(1+mx)} = \underset{x\to 0}{Lt} \frac{m(mx-1)}{(x^2+m^2)(1+mx)}$$

$$= \frac{-m}{m^2} = \frac{-1}{m} \text{ ——— } (1)$$

In (1)
$\therefore$ As 'm' changes, the limit value also changes i.e. limit is not unique. So, the function is discontinous. (or) not continous at origin.

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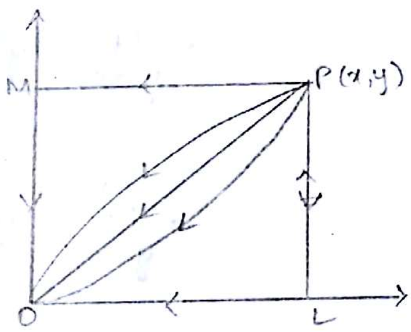
(3) $f(z) = \cancel{x^2y}\ \frac{2xy(x+y)}{x^2+y^2}$, if $(x,y) \neq (0,0)$

$= 0$, if $(x,y) = (0,0)$



Sol: Let $z \to 0$, along PLO

$$\underset{z\to 0}{Lt} f(z) = \underset{y\to 0}{Lt} \frac{2xy(x+y)}{x^2+y^2}$$

$$= \underset{x\to 0}{Lt}\ 0 = 0. \text{ ——— } (1)$$

Let $z \to 0$, along PMO

$$\underset{z\to 0}{Lt} f(z) = \underset{x\to 0}{Lt} \frac{2xy(x+y)}{x^2+y^2}$$

$$= \underset{y\to 0}{Lt}\ 0 = 0. \text{ ——— } (2)$$

Let $z \to 0$, along $y = mx$

$$\underset{z\to 0}{Lt} f(z) = \underset{x\to 0}{Lt} \frac{2x(mx)(x+mx)}{x^2+(mx)^2}$$

$$= \underset{x\to 0}{Lt} \frac{2mx^2(x+mx)}{x^2+m^2x^2} = \underset{x\to 0}{Lt} \frac{2x^3(m)(1+m)}{x^2(1+m^2)}$$

$$= \underset{x\to 0}{Lt} \frac{2x(m)(1+m)}{1+m^2} = 0. \text{ ——— } (3)$$

Let $z \to 0$, along $y = mx^2$.

$$\underset{x\to 0}{Lt}\ f(z) = \underset{x\to 0}{Lt}\ \frac{2x(mx^2)(x+mx^2)}{x^2+(mx^2)^2}$$

$$= \underset{x\to 0}{Lt}\ \frac{x^4(2m)(mx+1)}{x^2(1+m^2x^2)}$$

$$= \underset{x\to 0}{Lt}\ \frac{x^2(2m)(mx+1)}{(1+m^2x^2)}$$

$$= 0 \quad \text{———} \quad (4)$$

Let $z \to 0$, along $x = my^2$

$$\underset{x\to 0}{Lt}\ f(z) = \underset{y\to 0}{Lt}\ \frac{2y(my^2)(my^2+y)}{(my^2)^2+y^2}$$

$$= \underset{y\to 0}{Lt}\ \frac{y^4(2m)(my+1)}{y^2(m^2y^2+1)}$$

$$= \underset{y\to 0}{Lt}\ \frac{y^2(2m)(my+1)}{m^2y^2+1}$$

$$= 0 \quad \text{———} \quad (5)$$

$\therefore$ from (1), (2), (3), (4), (5), we can say that limit is unique.

$\therefore$ $f(z)$ is continous at zero.

** **Theorem:**

Necessary and sufficient conditions for $f(z)$ to be continous

The necessary and sufficient conditions for $w = f(z) = u(x,y) + iv(x,y)$ to be analytic in a region 'R' are

1) $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$ are continous functions of x & y in R.

(2) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (or) $u_x = v_y$ & $u_y = -v_x$

which are known as Cauchy - Riemann eq^n (or) C-R equations.

Proof:

Let $w = f(z) = u(x,y) + iv(x,y)$ —— (1) be analytic in a region 'R'.

To show that C-R equations are satisfied.

Given $f(z) = u + iv$

Let $\delta x, \delta y$ be the increments in x, y respectively.

Let $\delta u, \delta v, \delta z$ be the increments in u, v, z respectively.

$\delta z = \delta x + i\delta y$

Now, $f'(z) = ?$

$f(z)$ is analytic

$z = x + iy$

~~δx~~ $\rightarrow f(z+\delta z) = (u+\delta u) + i(v+\delta v)$

$$\underset{\delta z \to 0}{Lt} \frac{f(z+\delta z) - f(z)}{\delta z} = \underset{\delta z \to 0}{Lt} \frac{[(u+\delta u) + i(v+\delta v)] - (u+iv)}{\delta z}$$

$$= \underset{\delta z \to 0}{Lt} \frac{\cancel{u} + \delta u + \cancel{iv} + i\delta v - \cancel{u} - \cancel{iv}}{\delta z}$$

$$= \underset{\delta z \to 0}{Lt} \frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}$$

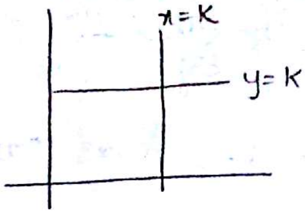
$$f'(z) = \frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z} \quad \text{—— (2)}$$

Since $w = f(z)$ is analytic in the region 'R', hence $f'(z)$ given by eq^n (2) should have a unique value in whatever manner

$\delta z \to 0$

Let $\delta z \to 0$ along a line $\parallel^{el}$ to x-axis.

$\delta y = 0$



$\delta z = \delta x + i\delta y = \delta x \; (\because \delta y = 0)$

$$f'(z) = \underset{\delta x \to 0}{Lt} \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{—} ③$$

Let $\delta z \to 0$ along a line $\parallel^{el}$ to Y-axis , $\delta x = 0$

$\delta z = \delta x + i\delta y = i\delta y \quad (\because \delta x = 0)$

$$f'(z) = \underset{\delta y \to 0}{Lt} \frac{\delta u}{i\delta y} + i\frac{\delta v}{i\delta y}$$

$$= \underset{\delta y \to 0}{Lt} \frac{i\delta u}{i^2\delta y} + \frac{\delta v}{\delta y}$$

$$f'(z) = \underset{\delta y \to 0}{Lt} -i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}$$

$$= -i \cancel{\frac{\delta y}{\delta y}} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \text{—} ④$$

from ③ & ④

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Equating real & img. part

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\boxed{u_x = v_y \quad \& \quad u_y = -v_x}$$

- Sufficient condition conversely supposse that $f(z)$ is any function satisfying the condition are continous in the region 'R' are $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ & $u_x = v_y$ & $u_y = -v_x$

To show $f'(z)$ exists.

From Taylor's theorem of two variables & neglecting second &

higher order terms in δx & δy we get

$$f(x+h, y+k) = f(x,y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{2!}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)^2 + \cdots$$

$$f(z+\delta z) = u(x+\delta x, y+\delta y) + i\,v(x+\delta x, y+\delta y)$$

$$= u(x,y) + \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + i\left[v(x,y) + \frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right]$$

from C-R eq^n.

$$f(z+\delta z) = f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y$$

$$f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(i^2\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)i\delta y$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right](\delta x + i\delta y)$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\delta z$$

$$\frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Therefore, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$\Rightarrow$ $f'(z)$ exists since $\frac{\partial u}{\partial x}$ & $\frac{\partial v}{\partial x}$ exist.

Therefore $f(z)$ is analytic.

Note :

The formula for $\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}$

$$= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

C-R eqⁿ in polar form:

Let (r, θ) be the polar co-ordinates of the point whose cartesian co-ordinates are x, y.

We have $x = r\cos\theta$, $y = r\sin\theta$

$$z = x + iy$$

$$z = r\cos\theta + i(r\sin\theta)$$

$$z = r(\cos\theta + i\sin\theta)$$

$$z = re^{i\theta}$$

$$w = f(z)$$

$$u + iv = f(re^{i\theta}) \quad \text{———} \quad (1)$$

Differenciating eqⁿ (1) partially w.r.t 'r'.

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})\, e^{i\theta}$$

Now, diff. eqⁿ (1) w.r.t 'θ'.

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \times r i e^{i\theta}$$

$$= ir\left[f'(re^{i\theta})e^{i\theta}\right]$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = ir\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]$$

$$= ir\frac{\partial u}{\partial r} + i^2 r\frac{\partial v}{\partial r}$$

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

Equating real & img. parts

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}, \quad \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

(or)

$$u_\theta = -rv_r, \quad v_\theta = ru_r$$

Q1) Show that f(z) =

i) $f(z) = z^2$

ii) $f(z) = z^3$

is analytic for all z.

Sol: $f(z) = (x+iy)^3$

$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$

$= (x^3 - 3xy^2) + i(3x^2y - y^3)$

$f(z) = u + iv$

$u = x^3 - 3xy^2$, $v = 3x^2y - y^3$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$, $\frac{\partial u}{\partial y} = \cancel{3x} - 6xy$.

$\frac{\partial v}{\partial x} = 6xy$, $\frac{\partial v}{\partial y} = 3x^2 - 3y^2$

$u_x = v_y$ & $u_y = -v_x$

∴ C-R equations are satisfied & partial derivatives exist.

∴ f(z) is analytic.

(i) $f(z) = z^2$.

Sol: $f(z) = (x+iy)^2$

$= x^2 + 2x(iy) + (iy)^2$

$= x^2 + 2x(iy) - y^2 = (x^2 - y^2) + i(2xy)$

$f(z) = u + iv$

$u = (x^2 - y^2)$ $v = 2xy$

$\frac{\partial u}{\partial x} = 2x$; $\frac{\partial u}{\partial y} = -2y$ $\frac{\partial v}{\partial x} = 2y$; $\frac{\partial v}{\partial y} = 2x$

$u_x = v_y$; $u_y = -v_y$

∴ C-R eq^n are satisfied & partial derivatives exist.

∴ f(z) is analytic.

2) i) $f(z) = z + 2\bar{z}$

ii) $f(z) = \sin x \sin y - i \cos x \cos y$.

iii) $f(z) = \dfrac{x - iy}{x^2+y^2}$

Check whether the following functions are analytic (or) not.

Sol: (iii) $f(z) = \dfrac{x}{x^2+y^2} + i\left(\dfrac{-y}{x^2+y^2}\right)$

$u = \dfrac{x}{x^2+y^2}$, $v = \dfrac{-y}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = -\left[\frac{(x^2+y^2)(0) - y(2x)}{(x^2+y^2)^2}\right] = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = -\left[\frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}\right] = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$u_x = v_y$ & $u_y = -v_x$

$\therefore$ C-R eq^n are satisfied & partial derivative exist.

$\therefore f(z)$ is analytic.

ii) $\sin x \sin y - i \cos x \cos y$.

$u = \sin x \sin y$, $v = -\cos x \cos y$.

$\dfrac{\partial u}{\partial x} = \cos x \sin y$; $\dfrac{\partial u}{\partial y} = \sin x \cos y$

$\dfrac{\partial v}{\partial x} = \sin x \cos y$; $\dfrac{\partial v}{\partial y} = \cos x \sin y$.

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

$\because$ C-R eq^n are not satisfied

$\therefore$ f(z) is not analytic.

i) $f(z) = z + 2\bar{z}$

$z = x + iy$

$f(z) = (x+iy) + 2(x-iy)$

$= (x+2x) + i\cancel{}y - 2iy.$

$= (\cancel{x+}3x) + i(-y)$

$f(z) = u + iv$

$u = 3x$; $v = -y$

$\frac{\partial u}{\partial x} = 3$, $\frac{\partial u}{\partial y} = 0$; $\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = -1$

$U_x \neq V_y$ & $U_y \neq -V_x$

$\therefore$ C-R eq^n are not satisfied.

$\therefore$ f(z) is not analytic.

3) Find where the following func ~~is~~ ceases (or) fails to be analytic.

i) $w = \frac{1}{z}$

ii) $w = \frac{z}{z-1}$

iii) $f(z) = \frac{z+2}{z(z^2+1)}$

Sol(iii): $f(z) = \frac{z+2}{z(z^2+1)}$

$= \frac{z+2}{z^3+z}$

$f'(z) = \frac{(z^3+z)(1) - (z+2)(3z^2+1)}{z^2(z^2+1)^2}$

$$= \frac{z^3 + z - 3z^3 - z - 6z^2 - 2}{z^2(z^2+1)^2}$$

$$= \frac{-2z^3 - 6z^2 - 2}{z^2(z^2+1)^2}$$

$$z^2(z^2+1)^2 = 0$$

$$z^2 = 0, \quad z^2+1 = 0$$

$$z = 0, \quad z^2 = -1$$

$$z = \pm i.$$

$\therefore$ $z = 0, +i, -i$ are the three singular points.

ii) $f(z) = \dfrac{z}{z-1}$

$\cancel{f'(z) = z-1)}$

$$f'(z) = \frac{(z-1)(1) - (z)(1)}{(z-1)^2}$$

$$f'(z) = \frac{\cancel{z} - 1 - \cancel{z}}{(z-1)^2}$$

$$f'(z) = \frac{-1}{(z-1)^2}$$

when $z = 1$, the function $f'(z)$ does not exist.

$\therefore$ 1 is the singular point.

i) $f(z) = \dfrac{1}{z}$

$$f'(z) = \frac{-1}{z^2}$$

when $z = 0$, the function $f'(z)$ does not exist.

$\therefore$ 0 is the singular point.

4) find all values of 'k' for $f(z) = e^{kx}(\cos ky + i \sin ky)$ is analytic.

ii) find p such that the function $f(z) = \frac{1}{2}\log(x^2+y^2) + i\left(\tan^{-1}\left(\frac{px}{y}\right)\right)$

is analytic.

Sol:ii) Given f(z) is analytic $\Rightarrow$ C-R eq^n are satisfied

Given $f(z) = \frac{1}{2}\log(x^2+y^2) + i\left(\tan^{-1}\left(\frac{px}{y}\right)\right)$

$u = \frac{1}{2}\log(x^2+y^2)$, $v = \tan^{-1}\left(\frac{px}{y}\right)$

$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$; $\frac{\partial u}{\partial y} = \frac{1}{2} \cdot \frac{1}{x^2+y^2} \cdot (2y) = \frac{y}{x^2+y^2}$

$\frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{px}{y}\right)^2}\left(\frac{p}{y}\right)$; $\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{px}{y}\right)^2} \times px\left(\frac{-1}{y^2}\right)$

$= \frac{y^2}{y^2+p^2x^2} \cdot \left(\frac{p}{y}\right) = \frac{py}{y^2+p^2x^2}$ $\qquad = \frac{y^2}{y^2+p^2x^2} \times \frac{-px}{y^2} = \frac{-px}{y^2+p^2x^2}$

Given, C-R eq^n are satisfied $\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\frac{x}{x^2+y^2} = \frac{-px}{p^2x^2+y^2} \quad ; \quad \frac{y}{x^2+y^2} = -\frac{py}{p^2x^2+y^2}$$

$$\boxed{p=-1.} \qquad \boxed{p=-1}$$

$\therefore$ For $f(z)$ is analytic for $p=-1$.

i) $f(z) = e^{x}(\cos ky + i\sin ky)$

Given, $f(z)$ is analytic so C-R eq^n are satisfied.

$$f(z) = e^{x}\cos ky + i(e^{x}\sin ky)$$

$$u = e^{x}\cos ky \quad ; \quad v = e^{x}\sin ky.$$

$$\frac{\partial u}{\partial x} = e^{x}\cos ky \qquad \frac{\partial v}{\partial x} = e^{x}\sin ky$$

$$\frac{\partial u}{\partial y} = -e^{x}k\sin ky \qquad \frac{\partial v}{\partial y} = e^{x}k\cos ky$$

Given C-R eq^n are satisfied

$$u_x = v_y \quad \& \quad u_y = -v_x$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$e^{x}\cos ky = e^{x}k\cos ky. \quad ; \quad \cancel{+}e^{x}k\sin ky = \cancel{+}e^{x}\sin ky$$

$$K = \frac{e^{x}\cos ky}{e^{x}\cos ky} \qquad K = \frac{e^{x}\sin ky}{e^{x}\sin ky}$$

$$\boxed{K=1} \qquad \boxed{K=1}$$

$\therefore$ $f(z)$ is analytic for $K=1$.

5) Prove that $f(z) = z^n$ (n is a +ve integer) is analytic & hence find its derivative.

Sol: ~~$z = x+iy$~~

~~$f(z) = (x+iy)^n$~~

Let (r,θ) be the polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$

polar form of complex number $z = re^{i\theta}$.

$$f(z) = z^n = (re^{i\theta})^n$$

$$f(z) = r^n e^{in\theta}$$

$$= r^n (\cos n\theta + i\sin n\theta)$$

$$= r^n \cos n\theta + i r^n \sin n\theta$$

$$u = r^n \cos n\theta \quad , \quad v = r^n \sin n\theta$$

$$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta \quad , \quad \frac{\partial u}{\partial \theta} = r^n(-n\sin n\theta)$$
$$= -nr^n \sin n\theta$$

$$\frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta \quad , \quad \frac{\partial v}{\partial \theta} - nr^n \cos n\theta$$

C.R. eqn in polar form,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}(nr^n \cos n\theta) = nr^{n-1}\cos n\theta = \frac{\partial u}{\partial r}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$= -r\frac{\partial v}{\partial r} = -r(nr^{n-1}\sin n\theta)$$

$$= -nr^n \sin n\theta = \frac{\partial u}{\partial \theta}$$

$$\therefore \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$\therefore$ C-R eqn is satisfied & partial derivative exists.

$\therefore$ $f(z)$ is analytic.

6) Show that $f(z) = \sin z$,

ii) $\cos z$ are analytic in the complex plane.

Sol) ii) $f(z) = \cos z$

$= \cos(x+iy)$

$= \cos x \cos(iy) - \sin x \sin(iy)$

complimentary funᶜ $\boxed{\cos(iy) = \cosh y \quad , \quad \sin(iy) = i\sinh y}$

$= \cos x \cosh y - \sin x i \sinh y$

$= u + iv$

$u = \cos x \cosh y \quad ; \quad v = -\sin x \sinh y$

$\frac{\partial u}{\partial x} = -\sin x \cosh y \quad ; \quad \frac{\partial v}{\partial x} = -\cos x \sinh y$

$\frac{\partial u}{\partial y} = \cos x (+\sinh y) \quad ; \quad \frac{\partial v}{\partial y} = -\sin x \cosh y$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ C-R eqⁿ are satisfied.

$\therefore$ $f(z)$ is analytic.

ii) $f(z) = \sin z$.

$= \sin(x+iy)$

$= \cancel{(\sin x \sin(iy) + \cos x \cos(iy))}$ x $\quad = \sin x \cos iy + \cos x \sin iy$

$= \cancel{i \sin x \sinh y + \cos x \cosh y} \quad = \sin x \cosh y + i \cos x \sinh y$

$= \cancel{\cos x \cosh y + i \sin x \sinh y}. \quad = u + iv$

$= \cancel{u + iv}$

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~~u = cosxcoshy~~ ; ~~v = sinxsinhy~~ $u = \sin x\cosh y$; $v = \cos x\sinh y$

~~$\frac{\partial u}{\partial x} = -\sin x\cosh y$~~ ; ~~$\frac{\partial v}{\partial x} = \cos x\sinh y$~~ $\frac{\partial u}{\partial x} = \cos x\cosh y$; $\frac{\partial v}{\partial x} = -\sin x\sinh y$

~~$\frac{\partial u}{\partial y} = \cos x\sinh y$~~ ; ~~$\frac{\partial v}{\partial y} = \sin x\cosh y$~~ $\frac{\partial u}{\partial y} = +\sin x\sinh y$; $\frac{\partial v}{\partial y} = \cos x\cosh y$

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$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ $f(z)$ is analytic.

Note: If a function $f(z)$ is analytic then it can be differenciated in the usual manner.

Ex: if $f(z) = \sin z$

$f'(z) = \cos z$.

- **Laplace equation in polar form:**

If $f''(z)$ exists then $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$ and $\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0$

Proof: let $f(z) = u(x,y) + iv(x,y)$ be an analytic function

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{——} \ (1) \quad , \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \quad \text{——} \ (2)$$

Differenciating eq^n (1) partially w.r.t 'r'.

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{r}\frac{\partial^2 v}{\partial r\partial \theta} + \frac{\partial v}{\partial \theta}\left(\frac{-1}{r^2}\right) \quad \text{——} \ (3)$$

Multiply eq^n (1) with $\frac{1}{r}$ on b.s.

$$\frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{r^2}\frac{\partial v}{\partial \theta} \quad \text{——} \ (4)$$

from (2),

$$+\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

$$\Rightarrow \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} = \frac{-\partial^2 v}{\partial \theta \partial r} \qquad \text{(diff. w.r.t '}\theta\text{')}$$

Multiply with $\frac{1}{r}$

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} \quad \text{——} \ (5)$$

(3) + (4) + (5)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \cancel{\frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta}} - \cancel{\frac{1}{r^2}\frac{\partial v}{\partial \theta}} + \cancel{\frac{1}{r^2}\frac{\partial v}{\partial \theta}} - \cancel{\frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r}}$$

($\because$ Assuming second order partial derivatives to be continous)

$\therefore$ Laplace eqn in polar form is $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

- Conjugate function:

If $f(z) = u(x,y) + iv(x,y)$ is analytic function then $u(x,y)$ & v are conjugate func. The relation between them is given by C-R eqn.

- Harmonic function:

Any func $\phi(x,y)$ which posseses continous partial derivatives of first & second order & satisfies laplace eqn i.e.

$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$ is called Harmonic function.

- Conjugate Harmonic function:

If a function $u(x,y)$ is harmonic in the domain and if we can find another Harmonic func $v(x,y)$ such that they satisfy C-R eqn and Laplace eqn then we say that $v(x,y)$ is the Harmonic conjugate of $u(x,y)$.

- Properties of analytic functions:

1) An analytic function with constant real part is constant.

2) An analytic function with constant Imaginary part is constant.

3) An analytic function with constant modulus is constant

4) The real and imaginary parts of an analytic function are harmonic.

Proof: Given $f(z)$ is analytic $\Rightarrow$ C-R eqⁿ are satisfied

C-R eqⁿ : $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ — ① , $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ — ②

Diff. eqⁿ ① w.r.t 'x' & eqⁿ ② w.r.t. 'y'.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \text{ — ③ ; } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y} \text{ — ④}$$

③ + ④ $\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad$ ($\because$ RHS is same)

$\therefore$ $\because$ u satisfies Laplace eqⁿ., u is harmonic function.

Diff. eqⁿ ① w.r.t 'y' & eqⁿ ② w.r.t 'x'.

$$\cancel{\frac{\partial^2 u}{\partial y^2} \neq \frac{\partial^2 v}{\partial y \partial x}}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} \text{ — ⑤ ; } \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial x^2} \text{ — ⑥}$$

⑤ − ⑥

$$\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2}$$

$$\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} = 0.$$

5) Orthogonal system:

Two families of curves $u(x,y) = c_1$ & $v(x,y) = c_2$ are said to form an orthogonal system if they intersect at right angles at each point of their intersection.

Property:

Every analytic funᶜ $f(z) = u + iv$ defines two families of curves $u(x,y) = c_1$ & $v(x,y) = c_2$ forming an orthogonal system.

Proof:

Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function.

Analytic functions $\Rightarrow$ C-R eq^n satisfied

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Let $u(x,y) = c_1$ — (1), $v(x,y) = c_2$ — (2)

Diff eq^n (1) w.r.t 'x'.

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} = 0$$

Let $m_1 = \frac{dy}{dx} = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$ — (3)

Diff. eq^n (2) w.r.t 'x'.

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \cdot \frac{dy}{dx} = 0$$

$$m_2 = \frac{dy}{dx} = \frac{-\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} \quad \text{— (4)}$$

$$m_1 m_2 = \frac{-\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} \times \frac{\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}} = -1$$

$\because m_1 m_2 = -1$, $u(x,y) = c_1$ & $v(x,y) = c_2$

$\therefore$ $u(x,y) = c_1$ & $v(x,y) = c_2$ form an orthogonal system.

6. If $f(x)$ & $g(x)$ are two polynomials of an analytic function then $f(x) \pm g(x)$, $f(x) \cdot g(x)$, $f(x)/g(x)$; $(g(x) \neq 0)$ are analytic.

7. If a function $f(z)$ is analytic then it is continous.

Q1) Prove that the func $f(z)$ defined by $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$, $z \neq 0$

$= 0$, $z = 0$

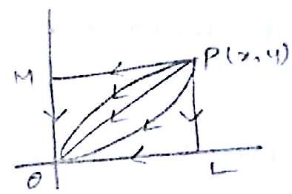
is continous & the C-R eqn are satisfied at the origin yet $f'(0)$ does not exist.

Sol: Given $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$

$$f(z) = \frac{(x^3-y^3) + i(x^3+y^3)}{x^2+y^2}$$

$$u = \frac{x^3-y^3}{x^2+y^2} \quad ; \quad v = \frac{x^3+y^3}{x^2+y^2}$$

Since u & v are polynomials hence they are continous at $z \neq 0$.



$$\lim_{z \to 0} f(z) = f(0)$$

let $z \to 0$ along PLO path ~~alon~~

along PL, y varies

along LO, x varies

$$\lim_{z\to 0} f(z) = \lim_{\substack{y \to 0 \\ x \to 0}} \frac{(x^3-y^3) + i(x^3+y^3)}{(x^2+y^2)} = \lim_{x \to 0} \frac{x^3 + ix^3}{x^2} = 0$$

Let $z \to 0$ along PMO path,

along PM, x varies

MO, y varies

$$\lim_{z\to 0} f(z) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{(x^3-y^3) + i(x^3+y^3)}{(x^2+y^2)}$$

$$\lim_{y \to 0} \frac{-y^3 + iy^3}{y^2} = 0$$

let $z \to 0$ along $y = mx$

$$\underset{z\to 0}{Lt}\, f(z) = \underset{x\to 0}{Lt} \frac{(x^3 - m^3x^3) + i\,(x^3 + m^3x^3)}{x^2 + m^2x^2}$$

$$= \underset{x\to 0}{Lt} \frac{x^3\,[(1-m^3) + i\,(1+m^3)]}{x^2\,(1+m^2)} = 0$$

Let $z \to 0$ along $y = mx^2$

$$\underset{z\to 0}{Lt}\, f(z) = \underset{x\to 0}{Lt} \frac{[x^3 - (mx^2)^3] + i\,[x^3 + (mx^2)^3]}{x^2 + (mx^2)^2}$$

$$= \underset{x\to 0}{Lt} \frac{x^3\,[(1-m^3x^3) + i\,(1+m^3x^3)]}{x^2\,(1+m^2x^2)} = 0$$

Along $x = my^2$ $\underset{z\to 0}{Lt}\, f(z) = 0$.

$\therefore$ $f(z)$ is continous. at whichever path $z \to 0$ & the limit is unique.

Path-II: Checking for C-R eq^n at origin.

$$u = \frac{x^3 - y^3}{x^2 + y^2}, \qquad v = \frac{x^3 + y^3}{x^2 + y^2}$$

$$\left.\frac{\partial u}{\partial x}\right)_{(a,b)} = \underset{x\to a}{Lt} \frac{u(x,b) - u(a,b)}{x - a}$$

formulas.

$$\left.\frac{\partial u}{\partial y}\right)_{(a,b)} = \underset{y\to b}{Lt} \frac{u(a,y) - u(a,b)}{y - b}$$

$$\left.\frac{\partial u}{\partial x}\right)_{(0,0)} = \underset{x\to 0}{Lt} \frac{u(x,0) - u(0,0)}{x - 0} = \underset{x\to 0}{Lt} \frac{x - 0}{x - 0} = 1.$$

$$\left.\frac{\partial u}{\partial y}\right)_{(0,0)} = \underset{y\to 0}{Lt} \frac{u(0,y) - u(0,0)}{y - 0} = \underset{y\to 0}{Lt} \frac{-y - 0}{y - 0} = -1$$

$$\left.\frac{\partial v}{\partial x}\right)_{(0,0)} = \underset{x\to 0}{Lt} \frac{v(x,0) - v(0,0)}{x - 0} = 1$$

$$\left.\frac{\partial v}{\partial y}\right)_{(0,0)} = \underset{y\to 0}{Lt} \frac{v(0,y)-v(0,0)}{y-0} = \underset{y\to 0}{Lt} \frac{y-0}{y-0} = 1$$

$\because U_x = V_y$ & $U_y = -V_x$.

$\therefore$ C-R eq^n are satisfied at origin.

Derivative at $z=0$:

$$\underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z-0}$$

let $z\to 0$ along $y=mx$.

$$\underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z-0} = \underset{x\to 0}{Lt} \frac{\dfrac{(x^3-m^3x^3)+i(x^3+m^3x^3)}{x^2+m^2x^2} - 0}{x+i(\underset{y}{mx}) - 0}$$

$(\because z=x+iy$
$y=mx)$

$$= \underset{x\to 0}{Lt} \frac{\cancel{x^3}[(1-m^3)+i(1+m^3)]}{\cancel{x^2}(1+m^2)\cancel{x}(1+im)} = \frac{(1-m^3)+i(1+m^3)}{(1+m^2)(1+im)}$$

As 'm' changes limit value changes i.e. limit is not unique
$\therefore$ $f'(z)$ does not exist at origin even though $f(z)$ is continous & C-R eq^n are satisfied at origin.

Therefore, $z=0$ is a singular point.

H.W.

(2) $f(z) = \sqrt{xy}$ is not analytic at origin even though C-R eq^n are satisfied.

Sol:

$$f(z) = \sqrt{xy} + 0i$$

$u = \sqrt{xy}$; $v = 0$

$\frac{\partial u}{\partial x} = \frac{1}{2\sqrt{x}} \cdot \sqrt{y}$; $\frac{\partial u}{\partial y} = \frac{1}{2\sqrt{y}} \cdot \sqrt{x}$

$\frac{\partial v}{\partial x} = 0$; $\frac{\partial v}{\partial y} = 0$

$\therefore$ C-R eq^n are not satisfied

3) If $w = \log z$ find $\frac{dw}{dz}$ and determine where w is non-analytic

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Sol:

$w = f(z) = u + iv$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$w = \log(x+iy)$

$w = \log(re^{i\theta})$ $\quad (\because z = re^{i\theta})$

$w = \log r + \log_e e^{i\theta}$

$w = \log r + i\theta = \log(\sqrt{x^2+y^2}) + i\theta$

$f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$

~~f(z)~~ $u = \frac{1}{2}\log(x^2+y^2)$, $v = \tan^{-1}(y/x)$

$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$

$\frac{\partial u}{\partial y} = \frac{1}{2} \cdot \frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$

$\frac{\partial v}{\partial x} = \frac{1}{1+(y/x)^2} \times y \times \frac{-1}{x^2} = \frac{-y}{x^2+y^2}$

$\frac{\partial v}{\partial y} = \frac{1}{(1+(y/x)^2)} \times \frac{1}{x} = \frac{x}{x^2+y^2}$.

$\therefore$ $u_x = v_y$ & $u_y = -v_x$

C-R eq^n are satisfied & the partial derivatives are continous except at origin.

Hence 'w' is analytic everywhere except at origin.

$\frac{dw}{dz} = f'(z) = \frac{1}{z}$ $(z \neq 0)$

$\therefore$ origin is the singular point.

3) ~~If $w = \log z$~~

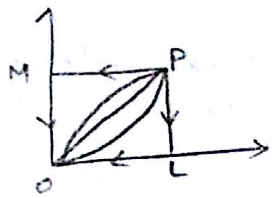
4) If $f(z) = \dfrac{x^3 y(y-ix)}{x^6+y^2}$, $z \neq 0$

$= 0$, $z = 0$

Prove that $\underset{z\to 0}{Lt} \dfrac{f(z)-f(0)}{z-0} \to 0$ along any radius vector

but not as $z \to 0$ along the curve $y = ax^3$.

Sol: let $z \to 0$ along PLO



Along PL, y varies

LO, x varies

$$\underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z-0} = \underset{\substack{y\to 0\\ x\to 0}}{Lt} \frac{\frac{x^3y(y-ix)}{x^6+y^2} - 0}{(x+iy)-0} = 0$$

Along PMO, $z \to 0$

Along PM, x varies

MO, y varies

$$\underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z-0} = \underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{\frac{x^3y(y-ix)}{x^6+y^2} - 0}{(x+iy)-0} = 0.$$

Along, $y = mx$, $z \to 0$.

$$\underset{z\to 0}{Lt} \frac{\frac{x^3(mx)(mx-ix)}{x^6+(mx)^2} - 0}{(x+i(mx))-0} = \underset{x\to 0}{Lt} \frac{x^5(m)(m-i)}{x^2(x^4+m^2)\,x(1+im)} = 0.$$

Let $z \to 0$, along $y = mx^2$

$$\underset{x\to 0}{Lt} \frac{\frac{x^3(mx^2)(mx^2-ix)}{x^6+(mx^2)^2} - 0}{(x+i(mx^2))-0} = \underset{x\to 0}{Lt} \frac{x^6(m)(mx-i)}{x^4(x^2+m^2)\,x(1+imx)} = 0$$

Let $z \to 0$, $y = \cancel{ax^3}\ mx^3$

$$\lim_{x \to 0} \frac{\dfrac{x^3(mx^3)(mx^3 - ix)}{x^6 + (mx^3)^2} - 0}{(x + imx^3) - 0} = \lim_{x \to 0} \frac{x^7 m(mx^2 - i)}{x^6(1+m^2)x(1+imx^2)}$$

$$= \lim_{x \to 0} \frac{m(mx^2 - i)}{(1+m^2)(1+imx^2)} = \frac{-im}{1+m^2}, \quad m \neq 0.$$

$\therefore$ $z \to 0$ along any radius vector but not along $y = ax^3$.

8/ Vd4 b2-4

5. Show that $f(z) = xy + iy$ is everywhere continous but not analytic.

Sol: u & v are polynomials and hence continous.

$\therefore$ $f(z)$ is also continous.

$u = xy$ $\qquad\qquad$ $v = y$

$\frac{\partial u}{\partial x} = y$ $\qquad\qquad$ $\frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = x$ $\qquad\qquad$ $\frac{\partial v}{\partial y} = 1$

$\because$ $u_x \neq v_y$; $u_y \neq -v_x$

6. If $f(z) = u + iv = \frac{1}{z}$ ST. $u(x,y) = c_1$, $v(x,y) = c_2$ intersect orthogonally.

Sol: $f(z) = \frac{1}{x+iy}$

$= \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$

$= \frac{x-iy}{x^2+y^2}$

$$u = \frac{x}{x^2+y^2} \quad ; \quad v = \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2)(0) - (-y)(2x)}{(x^2+y^2)^2} = \frac{\cancel{+x^2-y^2} + 2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\therefore$ $u_x = v_y$ & $u_y = -v_y$.

$\therefore$ C-R eq^n satisfies.

$\therefore$ $f(z)$ is analytic.

If $f(z)$ ~~is~~ $= u+iv$ is analytic then $u(x,y)=c_1$ & $v(x,y)=c_2$ are orthogonal to each other.

$\therefore$ $f(z) = u+iv$ where $u = \dfrac{x}{x^2+y^2}$ & $v = \dfrac{-y}{x^2+y^2}$ are orthogonal.

H.W

7. $f(z) = z^3$, prove that $u = c_1$, $v = c_2$ ~~both~~ cut each other orthogonally.

Sol:
$$f(z) = z^3$$
$$= (x+iy)^3$$
$$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$$
$$= (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$$u = x^3 - 3xy^2 \quad ; \quad v = 3x^2y - y^3$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad ; \quad \frac{\partial v}{\partial x} = 6xy$$

$$\frac{\partial u}{\partial y} = -6xy \quad ; \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$\therefore U_x = V_y$ & $U_y = -V_x$

$\therefore$ C-R eqⁿ satisfies

$\therefore f(z)$ is analytic.

If $f(z) = u+iv$ is analytic then $u(x,y) = C_1$, $v(x,y) = C_2$ are orthogonal to each other

$\therefore f(z) = u+iv$ where $u = x^3 - 3xy^2$, $v = 3x^2y - y^3$

3M

8. Prove that if $u = x^2 - y^2$, $v = \frac{-y}{x^2+y^2}$ both u & v satisfy Laplace equation but u+iv is not a regular function of z.

Sol: $u = x^2 - y^2$; $v = \frac{y}{x^2+y^2}$

$\frac{\partial u}{\partial x} = 2x$; $\frac{\partial v}{\partial x} = \frac{+2xy}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y} = -2y$; $\frac{\partial v}{\partial y} = \left[\frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}\right] = \frac{y^2-x^2}{(x^2+y^2)^2}$.

$\because U_x \neq V_y$ & $U_y \neq -V_x$

$\because$ C-R is not satisfied

$\therefore f(z)$ is not analytic

Diff. eqⁿ ① partially w.r.t 'x' & ② w.r.t 'y'.

$\frac{\partial^2 u}{\partial x^2} = 2$, $\frac{\partial^2 u}{\partial y^2} = -2$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\therefore u$ is Harmonic.

$\frac{\partial^2 v}{\partial x^2} = -2y\left[\frac{(x^2+y^2)(1) - x \times 2(x^2+y^2)(2x)}{(x^2+y^2)^4}\right] = \frac{2y(x^2+y^2)\left[x^2+y^2-4x^2\right]}{(x^2+y^2)^4}$

$= \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)^2(2y) - (y^2-x^2)\times 2(x^2+y^2)(2y)}{(x^2+y^2)^4}$$ (37)

$$= \frac{2y(x^2+y^2)\left[(x^2+y^2) - 2(y^2-x^2)\right]}{(x^2+y^2)^4}$$

$$= \frac{2y\left[x^2+y^2-2y^2+2x^2\right]}{(x^2+y^2)^3}$$

$$= \frac{2y(3x^2-y^2)}{(x^2+y^2)^3} = \frac{-2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\therefore$ v is harmonic

$\therefore$ f(z) is not analytic.

9. Show that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic.

Sol: $u = e^{-x}(x\sin y - y\cos y)$

$$\frac{\partial u}{\partial x} = e^{-x}(\sin y - \cancel{y\cos y}^{\,0}) + (x\sin y - y\cos y)e^{-x}(-1)$$

$$= e^{-x}\left[(\sin y - \cancel{y\cos y}^{\,0}) - (x\sin y - y\cos y)\right] = e^{-x}\left[\sin y - x\sin y + y\cos y\right]$$

$$\frac{\partial u}{\partial y} = e^{-x}\left[x(+\cos y) - (y(-\sin y) + \cos y)\right] + 0$$

$$= e^{-x}\left[+x\cos y + y\sin y - \cos y\right]$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-x}(0 - \sin y) + (\sin y - x\sin y)e^{-x}(-1) = -e^{-x}\left[\sin y + \sin y - x\sin y + y\cos y\right]$$

$$\frac{\partial^2 u}{\partial y^2} = \cancel{e^{-x}[-x\cos y + \cos y - x(\cos y)}\; e^{-x}\left[x(-\sin y) + [y(\cos y) + \sin y(1) - (-\sin y)\right]$$

$$= e^{-x}\left[-x\sin y + y\cos y + \sin y + \sin y\right]$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore$ u is harmonic.

10. Find k such that $f(x,y) = x^3 + 3kxy^2$ may be harmonic and find its conjugate.

Sol: $\frac{\partial u}{\partial x} = 3x^2 + 3ky^2$. $\frac{\partial u}{\partial y} = 6kxy$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$3x^2 + 3ky^2 = \frac{\partial v}{\partial y} \quad \text{— (1)}$$

I. b. s

$$3x^2\int dy + 3k\int y^2 dy = v$$

$$3x^2y + ky^3 + \phi(x) = v$$

$\cancel{3(2)xy}$ $3(2x)y + \phi'(x) = \frac{\partial v}{\partial x}$

$$u_y = -v_x$$

$$-6xy \mp \phi'(x) = 6kxy$$

$$\phi'(x) = 0 \quad , \quad \phi(x) = t$$

$$v(x,y) = 3x^2y - y^3 + t$$

$\frac{\partial^2 u}{\partial x^2} = 6x$, $\frac{\partial^2 u}{\partial y^2} = 6kx$

Given $f(z)$ harmonic , $6x + 6kx = 0$

$$\boxed{k = -1}$$

* SM

* 11. If $u(x,y) = x^3 - 3xy^2$ is harmonic and find the harmonic conjugate.

Sol: Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function.

$\therefore$ CR eq^n are satisfied.

$$U_x = V_y \quad \& \quad U_y = -V_x.$$

$u = x^3 - 3xy^2$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$; $\frac{\partial u}{\partial y} = -\cancel{3}\,6xy$.

$$\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3x^2 - 3y^2 \quad \text{— (1)}$$

Intergating (1) w.r.t 'y'.

$$\int \frac{\partial v}{\partial y} = 3x^2\int dy - 3\int y^2 dy$$

$$v = 3x^2y - \frac{3y^3}{3} + \phi(x)$$

(when integrated with y then constant in terms of x)

$$\frac{\partial v}{\partial x} = 6xy + \phi'(x)$$

$$\because \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$6xy + \phi'(x) = 6xy$$

$$\phi'(x) = 0$$

$$\phi(x) = K.$$

$$v(x,y) = 3x^2y - y^3 + K.$$

$$\therefore f(z) = u + iv = (x^3 - 3xy^2) + i(3x^2y - y^3 + K).$$

** 12. Find the analytic function whose real part is $e^{2x}(x\cos 2y - y\sin 2y)$.

Sol: Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function.

$\therefore$ C-R eq^n are satisfied

$$u_x = v_y \quad \& \quad u_y = -v_x$$

$$u = e^{2x}(x\cos 2y - y\sin 2y)$$

$$\frac{\partial u}{\partial x} = e^{2x}[\cos 2y - 0] + (x\cos 2y - y\sin 2y)(2e^{2x}).$$

$$= e^{2x}[\cos 2y + 2x\cos 2y - 2y\sin 2y]$$

$$\frac{\partial u}{\partial y} = e^{2x}[(-2\sin 2y)(x) - (y(2\cos 2y) + \sin 2y(1))] + 0$$

$$= e^{2x}[(-2x\sin 2y) - 2y\cos 2y - \sin 2y]$$

$$\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^{2x}[\cos 2y + 2x\cos 2y - 2y\sin 2y] \quad \text{—— (a)}$$

Integrate w.r.t. 'y'

$$\int \frac{\partial v}{\partial y} = e^{2x}\left[\int \cos 2y\,dy + 2x\int \cos 2y\,dy - 2\int y\sin 2y\right]$$

$$= e^{2x}\left[\frac{\sin 2y}{2} + \frac{2x\sin 2y}{2} - 2\left(y\left(\frac{-\cos 2y}{2}\right) + \int \frac{\cos 2y}{2}dy\right]\right.$$

$$= e^{2x}\left[\frac{\sin 2y}{2} + x\sin 2y + y\cos 2y - \frac{\sin 2y}{2}\right] + \phi(x)$$

$$= e^{2x}[x\sin 2y + y\cos 2y] + \phi(x) \quad \text{—— (2)}$$

diff. eq^n ⑤ w.r.t 'x'.

$$\frac{\partial v}{\partial x} = e^{2x}(\sin 2y + 0) + (x\sin 2y + y\cos 2y)(2e^{2x}) + \phi'(x)$$

$$= e^{2x}\left[\sin 2y + 2x\sin 2y + 2y\cos 2y\right] + \phi'(x) \quad \text{—— ⑤}$$

$$\because \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad (\text{from C-R eq}^n)$$

comparing ④ & ⑤

$$\phi'(x) = 0$$

$$\phi(x) = K$$

$$v = e^{2x}(x\sin 2y + y\cos 2y) + K.$$

$$\therefore f(z) = u + iv$$

$$f(z) = e^{2x}(x\cos 2y - y\sin 2y) + i e^{2x}(x\sin 2y + y\cos 2y)$$

$$= e^{2x}\left[x(\cos 2y + i\sin 2y) + i^2 y\sin 2y + iy\cos 2y\right]$$

$$= e^{2x}\left[x(\cos 2y + i\sin 2y) + iy((i\sin 2y) + \cos 2y)\right]$$

$$= e^{2x}\left[(x+iy)(\cos 2y + i\sin 2y)\right]$$

$$= e^{2x}(z)(e^{i2y})$$

$$= z e^{2(x+iy)}$$

$$f(z) = z e^{2z}.$$

Milne Thomson Method:

This is another method of finding analytic function $f(z)$ when u (or) v is given.

$$\because z = x + iy$$

$$\bar{z} = x - iy$$

$$x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2i}$$

Let $f(z) = u(x,y) + iv(x,y)$

$$f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2} + \frac{z-\bar{z}}{2i}\right) \quad \text{—— } (1)$$

Considering this as an identity in two independent variables z & $\bar{z}$ and put $z = \bar{z}$.

$$f(z) = u(z, 0) + iv(z, 0)$$

which is same as eq^n (1) when we replace x by z and y by 0.

(1) Find analytic function whose real part is given by $e^{2x}(x\cos 2y - y\sin 2y)$.

Sol: Let $f(z) = u(x,y) + iv(x,y)$ be an analytic fun^c.

$\therefore$ C-R eq^n are satisfied.

$$U_x = V_y \ \& \ U_y = -V_x$$

$$u = e^{2x}(x\cos 2y - y\sin 2y)$$

$$\frac{\partial u}{\partial x} = e^{2x}[\cos 2y + 2x\cos 2y - 2y\sin 2y]$$

$$\frac{\partial u}{\partial y} = e^{2x}[-2x\sin 2y - 2y\cos 2y - \sin 2y]$$

$$f(z) = u(x,y) + iv(x,y)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad (\because \text{CR eq}^n).$$

$$= e^{2x}[\cos 2y + 2x\cos 2y - 2y\sin 2y] + i[e^{2x}(+2x\sin 2y + 2y\cos 2y + \sin 2y)]$$

By Milne Thompson's method:

we replace x by z & y by 0.

$$f'(z) = e^{2z}(1+2z)$$

Integrating on both sides

$$f(z) = \int e^{2z}dz + 2\int ze^{2z}dz$$

$$= \frac{e^{2z}}{2} + 2\left[z\cdot\frac{e^{2z}}{2} - \int 1\cdot\frac{e^{2z}}{2}dz\right]$$

$\frac{d}{dx}$ 1st		$\int$ 2nd
		e^{2z}
z	+	$\frac{e^{2z}}{2}$
1	−	$\frac{e^{2z}}{4}$

$$= \frac{e^{2z}}{2} + ze^{2z} - \frac{e^{2z}}{2}$$

After $f(z)$ to find v open and write z as $x+iy$

$$f(z) = ze^{2z}$$

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2) Determine the analytic func whose real part is $\frac{x}{x^2+y^2}$.

Sol: Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function

$\therefore$ C-R eq^n satisfies.

$$U_x = V_y \text{ \& } U_y = -V_x$$

$$u = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$f(z) = u(x,y) + iv(x,y)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} + i\left(\frac{2xy}{(x^2+y^2)^2}\right)$$

By Milne Thompson Method,

Replace x by z, y by 0:

$$f'(z) = \frac{0^2 - z^2}{(z^2+0^2)^2} + i(0) = \frac{-z^2}{z^4} = \frac{-1}{z^2}$$

Integrating on b.s.

$$\int f(z) = \int \frac{-1}{z^2} dz = -\int \frac{1}{z^2} dz$$

$$\therefore f(z) = \frac{-1}{z}.$$

(13)

H.W

(2) $v = 3x^2y - y^3$

(3) $u = y + e^x\cos y$

(4) $v = \frac{x}{x^2+y^2}$

3. $v = 3x^2y - y^3$

Let $f(z) = u(x,y) + iv(x,y)$ be an analytic funⁿ.

$\therefore$ C-R eqⁿ satisfies.

$\therefore u_x = v_y$ & $u_y = -v_x$.

$v = 3x^2y - y^3$

$\frac{\partial v}{\partial x} = 6xy$; $\frac{\partial v}{\partial y} = 3x^2 - 3y^2$

$f(z) = u(x,y) + iv(x,y)$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$

$f'(z) = 3x^2 - 3y^2 + i(6xy)$

By Milne Thompson's method,

Replace x by z, y by 0.

$f'(z) = 3z^2 - 0 + i(0)$

$f'(z) = 3z^2$

I.B.S

$f(z) = 3\int z^2 dz = \cancel{3}\cdot\frac{z^3}{\cancel{3}}$

$f(z) = z^3$

(4) $u = y + e^x\cos y$.

Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function.

$\therefore$ C-R eqⁿ satisfies.

$\therefore u_x = v_y$ & $u_y = -v_x$

$u = y + e^x\cos y$

$\frac{\partial u}{\partial x} = e^x\cos y$, $\frac{\partial u}{\partial y} = 1 + e^x(-\sin y) = 1 - e^x\sin y$

$f(z) = u(x,y) + iv(x,y)$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$f'(z) = e^x\cos y - i(1 - e^x\sin y)$

By Milne Thompson method,

Replace x by z, y by 0.

$$f'(z) = e^0(\cos 0) - i(1 - e^0 \sin 0)$$

$$f'(z) = 1 - i(1) = 1 - i$$

I.B.S

$$\int f'(z) = \int (1-i)\,dz$$

$$f(z) = z - iz$$

(5) $v = \dfrac{x}{x^2+y^2}$

Let $f(z)$ be an analytic function.

$\therefore$ C-R eq^n satisfies.

$\therefore U_x = V_y$ & $U_y = -V_x$

$$v = \frac{x}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

$$f(z) = u(x,y) + iv(x,y)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{-2xy}{(x^2+y^2)^2} + i\left(\frac{y^2-x^2}{(x^2+y^2)^2}\right)$$

By Milne Thompson method,

Replace x by z & y by 0.

$$f'(z) = 0 + i\left(\frac{0-z^2}{(z^2+0^2)^2}\right)$$

$$f'(z) = i\left(\frac{-z^2}{z^4}\right) = i\left(-\frac{1}{z^2}\right)$$

I.B.S

$$\int f'(z) = i\int\left(\frac{-1}{z^2}\right)$$

$$f(z) = -i\frac{1}{z}.$$

(44)

Find $f(z)$ given

1) $u-v = (x-y)(x^2+4xy+y^2)$, $f(z)=u+iv$ is an analytic func, $z=x+iy$ (45)

Sol: $f(z)=u+iv$ is an analytic function $\Rightarrow$ C-R eq^n are satisfied.

$$u_x = v_y \ \& \ u_y = -v_x$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$f(z) = u+iv$$

$$\underline{if(z) = iu - v}$$

$$f(z)(1+i) = (u-v)+i(u+v) = F(z)$$

$$F(z) = (u-v)+i(u+v)$$

$$= U+iV$$

$$U = u-v, \quad V = u+v$$

$$\frac{\partial U}{\partial x} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \quad ; \quad \frac{\partial U}{\partial y} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}$$

$$\frac{\partial V}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \quad ; \quad \frac{\partial V}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\frac{\partial V}{\partial y} = -\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} = \frac{\partial U}{\partial x}$$

$$\therefore \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$

$$-\frac{\partial V}{\partial x} = -\left[\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\right] = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{\partial U}{\partial y}$$

$$\therefore \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

$\therefore$ C-R eq^n are satisfied for $F(z)$.

$F(z)$ is analytic.

$$F(z) = U+iV = (u-v)+i(u+v)$$

$$U = u-v = (x-y)(x^2+4xy+y^2)$$

$$\frac{\partial U}{\partial x} = (x-y)(2x+4y) + (x^2+4xy+y^2)(1)$$

$= 2x^2 + 4xy - 2xy - 4y^2 + x^2 + 4xy + y^2$

$= 3x^2 - 3y^2 + 6xy$

$$\frac{\partial U}{\partial y} = (x-y)(4x+2y) + (x^2 + 4xy + y^2)(-1)$$

$= 4x^2 + 2xy - 4xy - 2y^2 - x^2 - 4xy - y^2$

$= 3x^2 - 3y^2 - 6xy$

$F(z) = U + iV$

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$$

$= (3x^2 - 3y^2 + 6xy) - i(3x^2 - 3y^2 - 6xy)$

By Milne Thompson Method,

x by z & y by 0.

$F'(z) = 3z^2 - 3iz^2 = (3-3i)z^2$

I.B.S

$\int F'(z)\,dz = (3-3i)\int z^2 dz$

$$= (3-3i)\frac{z^3}{3} + c'$$

$F(z) = (1-i)z^3 + c'$

$$f(z) = \frac{F(z)}{1+i}$$

$$= \frac{1-i}{1+i} z^3 + c'$$

$$= \frac{(1-i)^2}{1-i^2} z^3 + c'$$

$$= \frac{1-1-2i}{2} z^3 + c'$$

$$f(z) = -iz^3 + c'$$

2) If $u-v = e^x(\cos y - \sin y)$, find $f(z) = u+iv$ which is an analytic function.

(47)

Sol: $f(z) = u+iv$ is an analytic function $\Rightarrow$ C-R eq^n are satisfied.

$$u_x = v_y \;\&\; u_y = -v_x$$

$3_1 + 2V$

$$f(z) = u+iv$$

$$if(z) = iu - v$$

$$f(z)(1+i) = (u-v)+i(u+v) = F(z) \quad \text{— (1)}$$

$$F(z) = (u-v)+i(u+v)$$

$$= U+iV$$

$$U = u-v \quad , \quad V = u+v$$

$$\frac{\partial U}{\partial x} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \quad ; \quad \frac{\partial V}{\partial x} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$$

$$\frac{\partial U}{\partial y} = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial V}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\frac{\partial V}{\partial y} = -\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} = \frac{\partial U}{\partial x}$$

$$\therefore \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$$

$$-\frac{\partial V}{\partial x} = -\left[\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\right] = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{\partial U}{\partial y}$$

$$\therefore \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

$\therefore$ C-R eq^n are satisfied for $F(z)$.

$F(z)$ is analytic.

$$F(z) = U+iV = (u-v)+i(u+v)$$

$$U = u-v = \cancel{(x-y)(x^2+4xy+y^2)}\; e^x(\cos y - \sin y)$$

$$\frac{\partial U}{\partial x} = \cancel{fx}\; e^x(\cos y - \sin y)$$

$$\frac{\partial U}{\partial y} = e^x(-\sin y - \cos y) = -e^x(\sin y + \cos y)$$

$F(z) = U + iV$

$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$

$= \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$

$= e^x(\cos y - \sin y) - i(-e^x(\sin y + \cos y))$

$= e^x(\cos y - \sin y) + i(e^x(\sin y + \cos y))$

By Milne Thompson method,

x by z, y by 0.

$F'(z) = e^z(\cos(0) - \sin(0)) + i(e^z(\sin(0) + \cos(0)))$

$= e^z(1-0) + i(e^z(0+1))$

$F'(z) = e^z + ie^z = e^z(1+i)$

I.B.S.

$\int F'(z) = \int e^z(1+i)\,dz.$

$= e^z + ie^z + c$

$F(z) = e^z(1+i) + c$

$f(z) = (1+i) = F(z)$ (from ①)

$f(z)(1+i) = e^z(1+i) + c'$

$\cancel{f(z) = e^z + c'}$

$f(z) = \frac{e^z(1+i)}{(1+i)} + c'$

$f(z) = e^z + c'$

3. If $2u+v = e^{2x}[(2x+y)\cos 2y + (x-2y)(\sin 2y)]$, find $f(z)$. 49

Sol: Let $f(z)$ be an analytic function $\Rightarrow$ C-R eq^n are satisfied.

$$u_x = v_y \quad \& \quad u_y = -v_x$$

$$f(z) = u + iv$$
$$-2i f(z) = -2iu + 2v$$
$$f(z)(1-2i) = (u+2v) - i(2u+v)$$

$$f(z) = u + iv$$
$$2i\, f(z) = 2iu - 2v$$
$$f(z)(1+2i) = (u-2v) + i(2u+v) = F(z) \quad \text{①}$$

$$F(z) = (u-2v) + i(2u+v)$$
$$= U + iV$$

$$U = u - 2v \quad ; \quad V = 2u + v$$

$$\frac{\partial U}{\partial x} = \frac{\partial u}{\partial x} - 2\frac{\partial v}{\partial x} \quad ; \quad \frac{\partial V}{\partial x} = 2\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$$

$$\frac{\partial U}{\partial y} = \frac{\partial u}{\partial y} - 2\frac{\partial v}{\partial y} \quad ; \quad \frac{\partial V}{\partial y} = 2\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\frac{\partial V}{\partial y} = -\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} = \frac{\partial U}{\partial x}$$

$$\therefore \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}.$$

$$-\frac{\partial V}{\partial x} = -\left[\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\right] = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = \frac{\partial U}{\partial y}$$

$$\therefore \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$$

$\therefore$ C-R eq^n are satisfied for $F(z)$.

$\therefore F(z)$ is analytic.

$$F(z) = (u-2v) + i(2u+v)$$

$$V = 2u+v = e^{2x}[(2x+y)\cos 2y + (x-2y)\sin 2y]$$

$$\frac{\partial V}{\partial x} = e^{2x}[\cancel{1+2}\,[(2x+y)\cos 2y + (x-2y)\sin 2y]\,e^{2x} + e^{2x}[2\cos 2y + \sin 2y]$$

$$\frac{\partial V}{\partial y} = e^{2x}[-2\sin 2y - 2(2\cos 2y)] = e^{2x}[-2\sin 2y - 4\cos 2y]$$

$$F(z) = U + iV$$

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x}$$

$$= \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x}$$

$$= e^{2x}[-2\sin 2y - 4\cos 2y] + i2[(2x+y)\cos 2y + (x-2y)\sin 2y] + e^{2x}$$
$$(2\cos 2y + \sin 2y)$$

By Milne Thompson method,

Replace x by z, y by 0.

$$F'(z) = e^{2z}[0-4] + i2[(2z+0) + (0)] + e^{2z}[2+0]$$

$$F'(z) = -4e^{2z} + i4z \overset{+2e^{2z}}{\wedge} = \cancel{4[iz - e^{2z}]}\ i4z + e^{2z}[-4+2]$$
$$= i4z - 2e^{2z}$$

I.B.S

$$\int F'(z) = \int 4[iz - e^{2z}]\,dz$$

$$F(z) = 2\left(i\frac{z^2}{2} - \frac{e^{2z}}{2}\right)$$

$$F(z) = [iz^2 - e^{2z}]$$

$$f(z)(1+2i) = [iz^2 - e^{2z}]$$

$$f(z) = \frac{[iz^2 - e^{2z}]}{(1+2i)}$$

4. Find the analytic function, given $v = \left(r - \frac{1}{r}\right)\sin\theta \quad (r \neq 0)$

Sol: Let $f(z)$ be analytic $\Rightarrow$ C-R eqⁿ satisfied.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad ; \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$$\frac{\partial v}{\partial r} = \left(1 + \frac{1}{r^2}\right)\sin\theta \; \text{—} ① \quad ; \quad \frac{\partial v}{\partial \theta} = \left(r - \frac{1}{r}\right)\cos\theta \; \text{—} ②$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\left(r - \frac{1}{r}\right)\cos\theta \quad \text{(from ②)}$$

$$\frac{\partial u}{\partial r} = \left(1 - \frac{1}{r^2}\right)\cos\theta$$

I.B.S w.r.t 'r'

$$\int \frac{\partial u}{\partial r}\, dr = \cos\theta \int \left(1 - \frac{1}{r^2}\right) dr$$

$$u = \cos\theta\left(r + \frac{1}{r}\right) + \phi(\theta) \quad \text{—— ③}$$

$$\frac{\partial u}{\partial \theta} = -\left(r + \frac{1}{r}\right)\sin\theta + \phi'(\theta) \quad ④$$

$$-r\frac{\partial v}{\partial r} = -r\left(1 + \frac{1}{r^2}\right)\sin\theta$$

$$= -\left(r + \frac{1}{r}\right)\sin\theta$$

$$\phi'(\theta) = 0 \Rightarrow \phi(\theta) = \text{constant } (K)$$

$$u = \left(r + \frac{1}{r}\right)\cos\theta + K.$$

$$f(z) = \left(r + \frac{1}{r}\right)\cos\theta + \left(r - \frac{1}{r}\right)\sin\theta.$$

★★ **Applications of analytic function to ~~slope~~ flow problems:**

Consider $w = f(z) = \phi(x,y) + i\,\psi(x,y)$

		$\phi(x,y)$	$\psi(x,y)$
1.	In fluid mechanics	velocity potential	Stream function
2.	Electrostatics & gravitational fields	Equipotential lines	Lines of force.
3.	Heat flow problems	Isothermals	Heat flow lines.
4.	Fluid mechanics	Potential function	Flux function.

1) If $w = \cancel{\phi + i\psi}\ \phi(x,y) + i\psi(x,y)$ represents the complex potential for an electric field and $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$. Determine the function ϕ. (52)

Sol: Let $w = \phi(x,y) + i\,\cancel{\phi}\,\psi(x,y)$ be an analytic function.

$\therefore$ CR eq^n are satisfied.

$$\phi_x = \psi_y \quad \& \quad \phi_y = -\psi_x$$

$$\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$$

$$\frac{\partial \psi}{\partial x} = 2x + \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = 2x + \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = 2x + \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\frac{\partial \psi}{\partial y} = -2y + \frac{(x^2+y^2)(0)\ \cancel{\phi} - x(2y)}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2+y^2)^2} = -\left(2y + \frac{2xy}{(x^2+y^2)^2}\right)$$

$$\cancel{\because \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = -2x + \frac{y^2 - x^2}{(x^2+y^2)^2}}$$

$$\because \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = -\left(2y + \frac{2xy}{(x^2+y^2)^2}\right) \quad \text{——} \ (1)$$

Integrating eq^n (1) w.r.t. 'x'

$$\int \frac{\partial \phi}{\partial x} = -\int 2y + \frac{2xy}{(x^2+y^2)^2}$$

$$\phi = -\left[2xy + 2y \int \frac{x}{(x^2+y^2)^2}\right]$$

$$\phi = -\left[2xy + 2y \cdot \frac{1}{2} \int \frac{2x}{(x^2+y^2)^2}\right]$$

$$\phi = -2xy + y\left[\frac{1}{x^2+y^2}\right] + c$$

$$\phi = -2xy + \frac{y}{x^2+y^2} + c$$

2) Show that if $f(z)$ is a regular function of z. Prove that

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 4|f'(z)|^2.$$

Proof: $f(z) = u+iv$

$$|f(z)| = \sqrt{u^2+v^2}$$

$$|f(z)|^2 = u^2+v^2 = \phi(u,v)$$

$$\frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial x}\right)\right] + 2\left[v\left(\frac{\partial^2 v}{\partial x^2}\right) + \left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial x}\right)\right]$$

$$= 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right] + 2\left[v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right] \quad \text{---} \; (1)$$

$$\frac{\partial^2 \phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right] + 2\left[v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right] \quad \text{---} \; (2)$$

(1) + (2)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2u\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] + 2v\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right] + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \quad \text{---} \; (3)$$

$\because f(z)$ is analytic, C-R eq^n u & v are harmonic func.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \& \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow \left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \rightarrow \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial x}\right)^2$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 2u(0) + 2v(0) + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$= 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \quad \text{---} \; (4)$$

$f(z) = u + iv$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ $(\because$ C-R eq$^n)$

$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$ —— (5)

subs (5) in (4)

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|f(z)|^2 = 4|f'(z)|^2$$

Hence proved.

3) If $f(z)$ is analytic then P.T. $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]|\text{Real part } f(z)|^2 = 2|f'(z)|^2$.

Sol: $f(z) = u + iv$; $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

$|f(z)| = \sqrt{u^2+v^2}$

$|f(z)|^2 = u^2 + v^2 = \phi(u,v)$

$\frac{\partial \phi}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}$; $\frac{\partial^2 \phi}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right] + 2\left[v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right]$

$\frac{\partial^2 \phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right] + 2\left[v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right]$

(1) + (2)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2\right] + 2\left[v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right] + 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2\right] + 2\left[v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

$\because f(z)$ is analytic, u & v are harmonic fun^c.

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ & $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \longrightarrow \left(\frac{\partial u}{\partial x}\right)^2 = \left(\frac{\partial v}{\partial y}\right)^2$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \longrightarrow \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial x}\right)^2$

Since u is real part of $f(z)$, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right].$$

$$= 2(u_x^2 + u_y^2)$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] |\text{Real part } f(z)|^2 = 2|f'(z)|^2$$

$$\therefore \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] |\text{Real part } f(z)|^2 = 2|f'(z)|^2$$

This year

** 2) If $f(z)$ is analytic with constant modulus. Show that $f(z)$ is constant function.

Sol: Given $f(z)$ is analytic $\Rightarrow$ C-R eq^n are satisfied

$$U_x = V_y \; \& \; U_y = -V_x$$

Given: $|f(z)|^2 = u^2 + v^2 = C$ —— (1)

To show that $u = c_1$ & $v = c_2$. , $c \neq 0$.

Diff. eq^n (1) w.r.t. 'x'.

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$$

$$u\frac{\partial u}{\partial x} = -v\frac{\partial v}{\partial x}$$

$$u\frac{\partial u}{\partial x} = v\left(\frac{\partial u}{\partial y}\right) \quad \text{(from C-R eq}^n\text{)}$$

$$\left(\frac{u}{v}\right)\frac{\partial u}{\partial x} = \cancel{\left(\frac{v}{v}\right)}\left(\frac{\partial u}{\partial y}\right) \quad \text{—— (2)}$$

Diff. eq^n (1) w.r.t 'y'.

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

$$2u\left(\frac{u}{v}\frac{\partial u}{\partial x}\right) + 2v\left(\frac{\partial u}{\partial x}\right) = 0$$

$$u^2\frac{\partial u}{\partial x} + v^2\frac{\partial u}{\partial x} = 0 \quad \text{(LCM)}$$

$$\frac{\partial u}{\partial x}(u^2+v^2) = 0$$

$\because u^2+v^2 \neq 0$, $\frac{\partial u}{\partial x} = 0 \Rightarrow u(x,y) = c_1$ (constant)

III'ly $v(x,y) = c_1$

$\therefore$ ~~v(x,y)~~ $f(z)$ is a constant func.

Case II: $c = 0$, $u^2+v^2=0$

$u=0$, $v=0$

$\therefore f(z)$ is a constant function.

5) If $w=f(z)$ is an analytic function of 'z' such that $f'(z) \neq 0$, prove that $\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right] \log|f'(z)| = 0$

Sol: $f(z)$ is analytic $\Rightarrow$ C-R eqn are satisfied.

$$\log|f'(z)| = \frac{1}{2}\log|f'(z)|^2 \qquad \text{(2 \& 2 gets cancelled)}$$

$$= \frac{1}{2}\log\left[f'(z)\cdot f'(\bar{z})\right] \qquad \left[\because |z|^2 = z\bar{z}\right]$$

$$= \frac{1}{2}\log\left[f'(z)\right] + \frac{1}{2}\log\left[f'(\bar{z})\right] = \phi(z) \quad \text{—— (1)}$$

To prove $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$

Diff. (1) w.r.t 'x'

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$\phi - z <^{x}_{y}$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial\phi}{\partial z}\cdot\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}\cdot\frac{\partial^2\phi}{\partial z^2}\cdot\frac{\partial z}{\partial x} \quad \text{— (1)}$$

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\partial\phi}{\partial z}\cdot\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y}\cdot\frac{\partial^2\phi}{\partial z^2}\cdot\frac{\partial z}{\partial y} \quad \text{—— (2)}$$

(1) + (2)

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{\partial\phi}{\partial z}\left[\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right] + \frac{\partial^2\phi}{\partial z^2}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] \quad \text{— (3)}$$

$$z = x + iy$$

$$\frac{\partial z}{\partial x} = 1 \quad ; \quad \frac{\partial^2 z}{\partial x^2} = 0$$

$$\frac{\partial z}{\partial y} = i \quad ; \quad \frac{\partial^2 z}{\partial y^2} = 0$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{—— } (4)$$

$$\frac{\partial z}{\partial x} = 1 \Rightarrow \left(\frac{\partial z}{\partial x}\right)^2 = 1$$

$$\frac{\partial z}{\partial y} = i \quad ; \quad \left(\frac{\partial z}{\partial y}\right)^2 = i^2 = -1 \quad \text{—— } (5)$$

sub^s (4) & (5) in (3)

$$LHS = \frac{\partial \phi}{\partial z}(0) + \frac{\partial^2 \phi}{\partial z^2}(1-1) = 0$$

$$\therefore \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] \log |f'(z)| = 0.$$