A

Course File

On

Mathematics – II

(Linear Algebra and Differential Equations)

MA101BS

(R16)

Submitted by

T RAGHUPATHI

Assistant Professor,

In the department of

Humanities & Sciences

CMR ENGINEERING COLLEGE

(Affiliated to J.N.T.U, HYDERABAD)

Kandlakoya(v),Medchal -501 401

1.**Institute Vision & Mission :**

Vision of the institute

To be recognized as a premier institution in offering value based and futuristic quality technical education to meet the technological needs of the society.

Mission of the institute

To impart value based quality technical education through innovative teaching and learning methods.

To continuously produce employable technical graduates with advanced technical skills to meet the current and future technological needs of the society.

To prepare the graduates for higher learning with emphasis on academic and industrial research.

Mechanical department:

Vision of the Department

To be a center of excellence in offering value based and futuristic quality technical education in the field of mechanical engineering.

Mission of the Department

1. Impart quality technical education imbibed with values by providing state of the art laboratories and effective teaching and learning process.

2.Produce industry ready mechanical engineering graduates with advanced technical and lifelong learning skills.

3.Prepare graduates for higher learning and research in mechanical engineering and its allied areas.

PROGRAMME OUTCOMES:

Engineering knowledge: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.

1. Problem analysis: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
2. Design/development of solutions: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
3. Conduct investigations of complex problems: Use research based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
4. Modern tool usage: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
5. The engineer and society: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
6. Environment and sustainability: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
7. Ethics: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
8. Individual and team work: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
9. Communication: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
10. Project management and finance: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one’s own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.
11. Lifelong learning: Recognize the need for, and have the preparation and ability to engage in independent and lifelong learning in the broadest context of technological change

Course out comes

CO1	**Interpret** the concept of Laplace transforms
CO2	**Apply** Laplace transform techniques for solving DE's
CO3	**Evaluate** integrals using Beta and Gamma functions
CO4	**Determine** the multiple integrals and can apply these concepts to find areas, volumes , moment of inertia etc of regions on a plane or in space
CO5	**Demonstrate** an understanding of vector differentiation.
CO6	**Find** the line, surface and volume integrals and converting them From One to another

Syllabus copy
UNIT–I

Laplace Transforms: Laplace transforms of standard functions, Shifting theorems, derivatives and integrals, properties- Unit step function, Dirac's delta function, Periodic function, Inverse Laplace transforms, Convolution theorem (without proof).

Applications: Solving ordinary differential equations (initial value problems) using Laplace transforms.

UNIT-II

Beta and Gamma Functions: Beta and Gamma functions, properties, relation between Beta and Gamma functions, evaluation of integrals using Beta and Gamma functions.

Applications: Evaluation of integrals.

UNIT–III

Multiple Integrals: Double and triple integrals, Change of variables, Change of order of integration.

Applications: Finding areas, volumes& Center of gravity (evaluation using Beta and Gamma functions).

UNIT–IV

Vector Differentiation: Scalar and vector point functions, Gradient, Divergence, Curl and their physical and geometrical interpretation, Laplacian operator, Vector identities.

UNIT–V

Vector Integration: Line Integral, Work done, Potential function, area, surface and volume integrals, Vector integral theorems: Greens, Stokes and Gauss divergence theorems (without proof) and related problems.

Text Books:

1. Advanced Engineering Mathematics by R K Jain & S R K Iyengar, Narosa Publishers
2. Engineering Mathematics by Srimanthapal and Subodh C. Bhunia, Oxford Publishers

References:

1. Advanced Engineering Mathematics by Peter V. O. Neil, Cengage Learning Publishers.
2. Advanced Engineering Mathematics by Lawrence Turyn, CRC Press

LESSON PLAN:

UNIT NO	**UNIT NAME**	**SUB TOPICS**	**NO. Of Lectures Required**	**Suggested Books**	**Remarks**
I	Laplace transform	Defination of Laplace Transforms Laplace transform of some standardfunctions. Laplace Transform of unitstep function First shifting theorem Second shifting theorem Scale property Inverse Laplace Transforms Convolution theorem Solving ODE using Laplace Problems on inverse Laplace Transforms Revision of Laplace Transform	L1 L2,L3,L4 L5,L6,L7 L8 L9,L10,L11 L12,L13, L14, L15, L16	T1, R1, R2	Unit-1 is completed by L16
II	Beta and Gamma functions	Defination of gamma functions Problems on gamma functions Definition of Beta functions Problems on Beta functions Beta and gamma relation Problems on beta and gamma relation. Revision	,L17, L18,L19, L20,L21, L22,L23, L24, L25,L26, L27, L28	T1,T2, R1, R2	Unit-2 is completed by L-28
III	**Multiple integrals**	1 Multiple integrals – double integrals. 2 Finding the area of a region using double integration 3. change of order of integration 4. change of variables (polar, cylindrical and spherical	L29, L30, L31 L32 L33,L34 L35,L36, L37,L38, L39, L40, L41,	T1, R1, R2	Unit-3 is completed by L-41
IV	**Vector Differentiation**	1.Introduction 2.Gradient, Divergence, Curl and their properties 3.Problems on Gradiant 4.problems on Divergence 5.Problems on curl	L42,L43, L44, L45, L46,L47, L48,L49, L50, L51,L52, L53,	T1,T2, R1, R2	Unit-4 is completed by L-53
V	**Vector Calculus**	Laplacian operator Line Integral – work done Surface Integral Volume Integral Green's ,Gauss's divergence and Stoke's theorem	L54,L55,L56,L57,L58,L59,L60, L61,L62,L63,L64.	T2	Unit-5 is completed by L-64
		TOTAL NO. OF CLASSES		64	

SESSION EXECUTION LOG

UNIT NO	UNIT NAME	SUB TOPICS	Expected period
I	**Laplace transform**	Defination of Laplace Transforms Laplace transform of some standardfunctions. Laplace Transform of unitstep function First shifting theorem Second shifting theorem Scale property Inverse Laplace Transforms Convolution theorem Solving ODE using Laplace Problems on inverse Laplace Transforms Revision of Laplace Transforms	27/07/17 to 26/08/17
II	**Beta and Gamma functions**	Defination of gamma functions Problems on gamma functions Definition of Beta functions Problems on Beta functions Beta and gamma relation Problems on beta and gamma relation. Revision	26/08/17 to 15/09/17
III	**Multiple integrals**	1 Multiple integrals – double integrals. 2 Finding the area of a region using double integration 3. change of order of integration 4. change of variables (polar, cylindrical and spherical	16/09/17 to 20/10/17
IV	**Vector Differentiation**	1.Introduction 2.Gradient, Divergence, Curl and their properties 3.Problems on Gradiant 4.problems on Divergence 5.Problems on curl	21/10/17 to 06/11/17
V	**Vector Calculus**	Laplacian operator Line Integral – work done Surface Integral Volume Integral Green's ,Gauss's divergence and Stoke's theorem	07/11/17 To 18/11/17
		TOTAL NO. OF CLASSES	64

UNIT-I

LAPLACE TRANSFORMS

Definition:

Let f(t) be a function defined for all positive values of t, then the Laplace transform of f(t) denoted by L{f(t)} or $\bar{f}$(s) is defined by

L{f(t)} = $\bar{f}$(s)= $\int_0^\infty e^{-st} f(t)$ dt. ----------------------1

Provided that the integral exists. Here the parameter ‘s’ is a real or complex number.

The relation (1) can also be written as f(t)= L^{-1}($\bar{f}$(s)).

f(t) is said to be inverse laplace transform of $\bar{f}$(s).

The symbol ‘L’ is called the laplace transform operator. The function f(t) must satisfy the following conditions for the existence of the laplace transform.

(a) The function f(t) must be piece-wise continuous in any limited interval $0<a\le t\le b$.

(b) The function f(t) is of exponential order.

Formulae

$L(1) = \frac{1}{s}$

$L(t) = \frac{1}{s^2}$

$L(t^n) = \frac{n!}{s^{n+1}}$

$L(e^{at}) = \frac{1}{s-a}$ (s-a>0)

$L(e^{-at}) = \frac{1}{s+a}$

$L(\cos at) = \frac{s}{s^2+a^2}$ if s>0

$L(\cosh at) = \frac{s}{s^2 + a^2}$

$L(\sin at) = \frac{a}{s^2+a^2}$ if s>0

$L(\sinh at) = \frac{a}{s^2-a^2}$

<u>First shifting theorem:</u>

If $L(f(t)) = \bar{f}(s)$ then $L(e^{at} f(t) = \bar{f}(s-a)$, s-a>0

Second shifting theorem:

If $L(f(t)) = \bar{f}(s)$ and $g(t) = \begin{cases} f(t-a) & t>a \\ 0 & t<a \end{cases}$ then

$$L(g(t))= e^{-as}\bar{f}(s)$$

1. Find the laplace transform of e^{-3t} (2cos5t – 3sin5t).

   We have

   $L(2\cos 5t - 3\sin 5t) = \frac{2s}{s^2+25} - \frac{3(5)}{s^2+25} = \frac{2s-15}{s^2+25}$

   Now applying first shifting theorem

   $L(e^{-3t}(2\cos 5t - 3\sin 5t)) = \frac{2s-15}{s^2+25}$

   Changes s to s+3

   $$= \frac{2(s+3)-15}{(s+3)^2+25} = \frac{2s-9}{s^2+6s+34}$$

2. If $L(f(t)) = \frac{9s^2-12s+15}{(s-1)^3}$ then find L(f(3t)) using change of scale

   property.

   Given $L(f(t)) = \frac{9s^2-12s+15}{(s-1)^3} = \bar{f}(s)$

   By change of scale property

   $L(f(3t)) = \frac{1}{3}\bar{f}(\frac{s}{3})$

   $$= \frac{1}{3}\frac{9(\frac{s}{3})^2-12(\frac{s}{3})+5}{(\frac{s}{3}-1)^3} = \frac{9(s^2-4s+5)}{(s-3)^3}$$

3. Find the laplace transform of g(t) where g(t) = cos $(t-\frac{\pi}{3})$ if $t>\frac{\pi}{3}$

0 if $t<\frac{\pi}{3}$

Sol: Let f(t) = cost

$$L(f(t)) = \frac{s}{s^2+1} = \bar{f}(s)$$

$$g(t) = \begin{cases} f(t-\frac{\pi}{3}) = \cos(t-\frac{\pi}{3}), & \text{if } t>\frac{\pi}{3} \\ 0 & \text{if } t<\frac{\pi}{3} \end{cases}$$

Applying second shifting theorem, we get

$$L(g(t)) = e^{-\frac{\pi}{3}(s)}(\frac{s}{s^2+1}) = \frac{se^{-\frac{\pi}{3}(s)}}{s^2+1}$$

4. Find $L(t^2 e^{-2t}\cos 2t)$

$$L(\cos t) = \frac{s}{s^2+1}$$

$L(e^{-2t}\cos t) = \frac{s}{s^2+1}$ changing s to s+2

$$= \frac{s+2}{(s+2)^2+1}$$

$$= \frac{s+2}{s^2+4s+5}$$

$$L(t^2e^{-2t}\cos t) = (-1)^2 \frac{d^2}{ds^2}(\frac{s+2}{s^2+4s+5})$$

$$= (\frac{(2s+4)(s^2+4s+1)}{(s^2+4s+5)^3})$$

5. using laplace transform evaluate $\int_0^\infty \frac{\cos at-\cos bt}{t}\,dt$

Sol: Given integral is same as $\int_0^\infty e^{-st}\frac{(\cos at-\cos bt)}{t}\,dt$

i.e $L(\frac{(\cos at-\cos bt)}{t})$ where s = 0

since $L(\cos at-\cos bt) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$$L(\frac{(\cos at-\cos bt)}{t}) = \int_s^\infty (\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2})\,ds$$

$$= \frac{1}{2}\int_s^\infty (\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2})\,ds$$

$$=\frac{1}{2}[\log(s^2+a^2)-\log(s^2+b^2)]_s^{\infty}$$

$$=\frac{1}{2}[\log(\frac{s^2+a^2}{s^2+b^2})]_s^{\infty}$$

$$=\frac{1}{2}[\log(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}})]_s^{\infty}$$

$$=\frac{1}{2}[\log 1-\log(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}})]$$

$$=\frac{1}{2}[0-\log(\frac{s^2+a^2}{s^2+b^2})]$$

$$=-\frac{1}{2}\log(\frac{s^2+a^2}{s^2+b^2})$$

$$\int_0^{\infty} e^{-st}\frac{(cosat-cosbt)}{t}\,dt=\frac{1}{2}\log(\frac{s^2+b^2}{s^2+a^2})$$

Take s=0 then

$$\int_0^{\infty}\frac{cosat-cosbt}{t}\,dt=\frac{1}{2}\log(\frac{b}{a})^2=\log(\frac{b}{a})$$

6.Find $L(\frac{e^{-t}sint}{t})$

we know that $L(\text{sint})=\frac{1}{s^2+1}$

by first shifting theorem $L(e^{-t}\text{sint})=(\frac{1}{s^2+1})_{s\to s+1}$

$$=\frac{1}{(s+1)^2+1}$$

$$=\bar{f}(s)$$

$$L(\frac{e^{-t}sint}{t})=\int_s^{\infty}\bar{f}(s)\,ds$$

$$=\int_s^{\infty}\frac{1}{(s+1)^2+1}\,ds=[tan^{-1}(s+1)]_s^{\infty}$$

$$=tan^{-1}\infty-tan^{-1}(s+1)$$

$$=\frac{\pi}{2}-tan^{-1}(s+1)$$

$$=cot^{-1}(s+1)$$

$$\left[\because tan^{-1}x+cot^{-1}x=\frac{\pi}{2}\right]$$

7. Find the laplace transform of periodic function f(t) with period T, where

$$f(t) = \begin{cases} \frac{4Et}{T} - E, & 0 \le t \le \frac{T}{2} \\ 3E - \frac{4Et}{T}, & \frac{T}{2} \le t \le T \end{cases}$$

Given $$f(t) = \begin{cases} \frac{4Et}{T} - E, & 0 \le t \le \frac{T}{2} \\ 3E - \frac{4Et}{T}, & \frac{T}{2} \le t \le T \end{cases}$$

Since f(t) is periodic function with period T, we have

$$L(f(t)) = \frac{1}{1-e^{-ST}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-ST}} \left[\int_0^{\frac{T}{2}} e^{-st} f(t)\, dt + \int_{\frac{T}{2}}^{T} e^{-st} f(t)\, dt\right]$$

$$= \frac{1}{1-e^{-ST}} \left[\int_0^{\frac{T}{2}} e^{-st} \left(\frac{4Et}{T} - E\right) dt + \int_{\frac{T}{2}}^{T} e^{-st} \left(3E - \frac{4Et}{T}\right) dt\right]$$

$$= \frac{1}{1-e^{-st}} \left[\frac{4E}{T} \int_0^{\frac{T}{2}} te^{-st} dt - E\int_0^{\frac{T}{2}} e^{-st}\, dt + 3E \int_{\frac{T}{2}}^{T} e^{-st}\, dt - \frac{4E}{T} \int_{\frac{T}{2}}^{T} te^{-st}\, dt\right.$$

$$= \frac{1}{1-e^{-ST}} \left[\frac{4E}{T} \left(t\left(\frac{e^{-st}}{-s}\right) - \frac{e^{-st}}{s^2}\right)_0^{T/2} - E\left(\left(\frac{e^{-st}}{-s}\right)\right)_0^{T/2} + 3E \left(\left(\frac{e^{-st}}{-s}\right)\right)_{T/2}^{T} - \frac{4E}{T} \left(t\left(\frac{e^{-st}}{-s}\right) - \frac{e^{-st}}{s^2}\right)_{T/2}^{T}\right.$$

$$= \frac{1}{1-e^{-ST}} \left[\frac{4E}{T} \left(\frac{-T}{2s} e^{\frac{-Ts}{2}} - \frac{e^{\frac{-Ts}{2}}}{s^2} + \frac{1}{s^2}\right) + \frac{E}{s}\left(e^{\frac{-Ts}{2}} - 1\right) - \frac{3E}{s}\left(e^{-st} - e^{\frac{-Ts}{2}}\right)\right]$$

$$= \frac{1}{1-e^{-ST}} \left[\frac{-2E}{s} e^{\frac{-Ts}{2}} + \frac{4E}{Ts^2}\left(1 - e^{\frac{-Ts}{2}}\right) + \frac{E}{s} e^{\frac{-Ts}{2}} - \frac{E}{s} - \frac{3E}{s} e^{-Ts} + \frac{3E}{s} e^{\frac{-Ts}{2}}\right]$$

$$= \frac{1}{1-e^{-ST}} \left[\frac{4E}{Ts^2}\left(1 - e^{\frac{-Ts}{2}}\right) - \frac{2E}{s} e^{\frac{-Ts}{2}} - \frac{3E}{s} e^{-Ts} - \frac{E}{s}\right]$$

$$= \frac{E}{s(1-e^{-sT})} \left[\frac{4}{Ts}\left(1 - e^{\frac{-Ts}{2}}\right) - 2e^{\frac{-Ts}{2}} - 3e^{-Ts} - 1\right.$$

8. Find $L\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3$

$(\sqrt{t}+\frac{1}{\sqrt{t}})^3 = (\sqrt{t})^3 + (\frac{1}{\sqrt{t}})^3 + 3.\ \sqrt{t}\ .\ \frac{1}{\sqrt{t}}\ (\sqrt{t}+\frac{1}{\sqrt{t}})$

$= t^{\frac{3}{2}} + \frac{1}{t^{\frac{3}{2}}} + 3\ (t^{\frac{1}{2}} + \frac{1}{t^{\frac{1}{2}}})$

$= t^{\frac{3}{2}} + t^{-\frac{3}{2}} + 3t^{\frac{1}{2}} + 3t^{-\frac{1}{2}}$

$L(t^{\frac{3}{2}} + t^{-\frac{3}{2}} + 3t^{\frac{1}{2}} + 3t^{-\frac{1}{2}}) = L(t^{\frac{3}{2}}) + L(t^{-\frac{3}{2}}) + 3L(t^{\frac{1}{2}}) + 3L(t^{-\frac{1}{2}}$

)

$L(t^n) = \frac{\sqrt{n+1}}{s^{n+1}}$ when n>-1

$L(t^{3/2}) = \frac{\sqrt{\frac{3}{2}+1}}{s^{\frac{3}{2}+1}} = \frac{\frac{3}{2}\sqrt{\frac{1}{2}+1}}{s^{\frac{5}{2}}} = \frac{\frac{3}{2}.\frac{1}{2}\sqrt{\Pi}}{s^{\frac{5}{2}}}$

$L(t^{-3/2}) = \frac{\sqrt{-\frac{3}{2}+1}}{s^{-\frac{3}{2}+1}} = \frac{\sqrt{-\frac{1}{2}}}{s^{-\frac{1}{2}}} = \frac{-2\sqrt{\Pi}}{s^{-\frac{1}{2}}}$ i.e. $\sqrt{-\frac{1}{2}} = -$

$2\sqrt{\Pi}$

$L(t^{1/2}) = \frac{\sqrt{\frac{1}{2}+1}}{s^{\frac{1}{2}+1}} = \frac{\frac{1}{2}\sqrt{\Pi}}{s^{\frac{1}{2}+1}}$

$L(t^{-1/2}) = \frac{\sqrt{-\frac{1}{2}+1}}{s^{-\frac{1}{2}+1}} = \frac{\sqrt{\Pi}}{s^{\frac{1}{2}}}$

$L(t^{\frac{3}{2}} + t^{-\frac{3}{2}} + 3t^{\frac{1}{2}} + 3t^{-\frac{1}{2}}) = \frac{3}{4}\ (\frac{\sqrt{\Pi}}{s^{\frac{5}{2}}})\ -\ \frac{2\sqrt{\Pi}}{s^{-\frac{1}{2}}} + \frac{3}{2}\ (\frac{\sqrt{\Pi}}{s^{\frac{3}{2}}}) + \frac{3\sqrt{\Pi}}{s^{\frac{1}{2}}}$

9. Using Laplace transform

show that $\int_0^\infty t^2 e^{-4t} \sin 2t\, dt = \frac{11}{500}$

sol: we note that the given integral is same as $\int_0^\infty t^2 e^{-st} \sin 2t$ dt

i.e. L($t^2 \sin 2t$) where s =4

But $L(t^2 \sin 2t) = (-1)^2 \frac{d^2}{ds^2} (L(\sin 2t))$

$= (-1)^2 \frac{d^2}{ds^2} (\frac{2}{(s)^2+4})$

$= 2.\ \frac{d}{ds} (\frac{-1}{(s^2+4)^2}\ 2s)$

$$= -4\frac{d}{ds}\left(\frac{s}{(s^2+4)^2}\right)$$

$$= -4\left[\frac{(s^2+4)^2.1 - s.2(s^2+4).2s}{(s^2+4)^4}\right]$$

$$= -4\left[\frac{(s^2+4)[(s^2+4)-4(s^2)]}{(s^2+4)^4}\right]$$

$$= \frac{-4(4-3s^2)}{(s^2+4)^3}$$

$$\int_0^\infty t^2 e^{-st} \sin 2t \, dt = \frac{-4(4-3s^2)}{(s^2+4)^3}$$

Putting s = 4, we get

$$\int_0^\infty t^2 e^{-4t} \sin 2t \, dt = \frac{4(48-4)}{(20)^3} = \frac{4(44)}{20(400)} = \frac{11}{500}$$

10. Find $L\left(\int_0^t \left(\int_0^t \cosh at \, dt\right) dt\right)$

Sol: Let f(t) = coshat then

$$L(f(t)) = L(\cosh at) = \frac{s}{(s^2-a^2)} = \bar{f}(s)$$

Using the theorem on Laplace transform of integral

$$L\left(\int_0^t \cosh at \, dt\right) = \frac{1}{s}\bar{f}(s)$$

$$= \frac{1}{s}.\frac{s}{(s^2-a^2)} = \frac{1}{(s^2-a^2)}$$

Applying again

$$L\left(\int_0^t \left(\int_0^t \cosh at \, dt\right) dt\right) = \frac{1}{s}.\frac{s}{(s^2-a^2)} = \frac{1}{s(s^2-a^2)}$$

11. Find $L^{-1}\left(\log\left(\frac{1+s}{s^2}\right)\right)$

Let $\bar{f}(s) = \log\left(\frac{1+s}{s^2}\right)$

$$= \log(1+s) - 2\log s$$

Then $\overline{(f}^{\,1}s) = \left(\frac{1}{1+s}\right) - \frac{2}{s}$

$$L^{-1}\,\overline{(f}^{\,1}s) = L^{-1}\left(\frac{1}{1+s}\right) - 2L^{-1}\left(\frac{1}{s}\right)$$

$$= e^{-t} - 2$$

$$(-1)\,t\,L^{-1}\,\overline{(f}^{\,1}s) = e^{-t} - 2$$

$$= L^{-1}\,\overline{(f}^{1}s) = \frac{2-e^{-t}}{t}$$

$$\therefore L^{-1}\left(\log\left(\frac{1+s}{s^2}\right)\right) = \frac{2-e^{-t}}{t}$$

INVERSE LAPLACE TRANSFORM:

If $\bar{f}(s)$ is the Laplace transform of a function f(t) then f(t) is called the inverse laplace transform of $\bar{f}(s)$ and it is denoted by $L^{-1}\{\bar{f}(s)\}$ i.e.

$$f(t) = L^{-1}\{\bar{f}(s)\}$$

where L^{-1} is called the inverse Laplace transform operator.

Table of Inverse Laplace transform:

S.no	$\bar{f}(s)$	$L^{-1}\{\bar{f}(s)\} = f(t)$
1	$\frac{1}{s}$	1
2	$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$
3	$\frac{1}{s^{n+1}}$, n > -1	$\frac{t^n}{\sqrt{(n+1)}}$
4	$\frac{1}{s-a}$	e^{at}
5	$\frac{1}{s+u}$	e^{-at}
6	$\frac{1}{s^2+a^2}$	$\frac{1}{a}$ sinat
7	$\frac{s}{s^2+a^2}$	cosat
8	$\frac{1}{s^2-a^2}$	$\frac{1}{a}$ sinhat
9	$\frac{s}{s^2-a^2}$	coshat
10	$\frac{1}{(s-a)^2+b^2}$ or	$\frac{1}{b}e^{at}$ sinbt or $\frac{1}{b}e^{-at}$

	$\frac{1}{(s+a)^2+b^2}$	sinbt
11	$\frac{s-a}{(s-a)^2-b^2}$ or $\frac{s+a}{(s+a)^2+b^2}$	e^{at}cosbt or e^{-at}cosbt
12	$\frac{1}{(s-a)^2-b^2}$ or $\frac{1}{(s+a)^2-b^2}$	$\frac{1}{b}e^{at}$ sinhbt or $\frac{1}{b}e^{-at}$ sinhbt
13	$\frac{s-a}{(s-a)^2-b^2}$ or $\frac{s+a}{(s+a)^2-b^2}$	e^{at}coshbt or e^{-at}coshbt
14	$\frac{2as}{(s^2+a^2)^2}$	t sinat
15	$\frac{s^2-a^2}{(s^2+a^2)^2}$	tcosat

12. Find $L^{-1}[\frac{s}{(s^2+1)(s^2+9)(s^2+25)}]$

Sol. We have $\frac{1}{(s^2+1)(s^2+9)} = \frac{1}{8}(\frac{1}{(s^2+1)} - \frac{1}{(s^2+9)})$

Resolving into partial fractions

$$\therefore \frac{1}{(s^2+1)(s^2+9)(s^2+25)} = \frac{1}{8}[\frac{1}{(s^2+1)(s^2+25)} - \frac{1}{(s^2+9)(s^2+25)}]$$

$$= \frac{1}{8}[\frac{1}{24}(\frac{1}{(s^2+1)} - \frac{1}{(s^2+25)}) - \frac{1}{16}(\frac{1}{(s^2+9)} - \frac{1}{(s^2+25)})$$

$$= \frac{1}{8}[\frac{1}{24}(\frac{1}{(s^2+1)}) - \frac{1}{16}(\frac{1}{(s^2+9)}) + \frac{1}{48}(\frac{1}{(s^2+25)})]$$

$$= \frac{1}{8*24*16}[(\frac{16}{(s^2+1)}) - (\frac{24}{(s^2+9)}) + (\frac{8}{(s^2+25)})]$$

$$\therefore \frac{1}{(s^2+1)(s^2+9)(s^2+25)} = \frac{1}{3092}[16.\frac{s}{(s^2+1)} - 24.\frac{s}{(s^2+3^2)} + 8.\frac{s}{(s^2+5^2)}]$$

Hence $L^{-1}[\frac{s}{(s^2+1)(s^2+9)(s^2+25)}] =$

$$= \frac{1}{3092}[16.L^{-1}\frac{s}{(s^2+1)} - 24.L^{-1}\frac{s}{(s^2+3^2)} + 8.L^{-1}\frac{s}{(s^2+5^2)}]$$

$$= \frac{1}{3092}[\ 16\text{cosat} - 24\text{cos3t} + 8\text{cos5t}]$$

13. Find $L^{-1}[\frac{e^{-2s}}{(s^2+4s+5)}]$

We have

$L^{-1}\left[\frac{1}{(s^2+4s+5)}\right] = L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$

$= e^{-2t} L^{-1}\left[\frac{1}{(s)^2+1}\right] = e^{-2t}$ sint = f(t)

∴ By second shifting theorem

$L^{-1}\left[\frac{e^{-2s}}{(s^2+4s+5)}\right] = \begin{cases} e^{-2(t-2)} \sin(t-2), & t>2 \\ 0 & t<2 \end{cases}$

(or)

$L^{-1}\left[\frac{e^{-2s}}{(s^2+4s+5)}\right] = e^{-2(t-2)}\sin(t-2)\, H(t-2)$

Where H(t-2) is the Heaviside unit step function.

14. Find inverse Laplace transform of $\log(\frac{s+1}{s-1})$

Sol: Let $\bar{f}(s) = \log(\frac{s+1}{s-1})$

$L(f(t)) = \log(\frac{s+1}{s-1}) = \log(s+1) - \log(s-1)$

$\therefore L(f(t)) = (-1)\frac{d}{ds}(\log(s+1) - \log(s-1))$

$= (-1)\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$

$= \frac{1}{s-1} - \frac{1}{s+1}$

$= L(e^{t} - e^{-t}) = L(2\sinh t)$

Comparing b.s we get

$L(f(t)) = 2\sinh t$

$\therefore f(t) = \frac{2}{t}\sinh t$

15. Find $L^{-1}\left[\frac{1}{s(s^2+1)(s^2-1)}\right]$

We have

$L^{-1}\left[\frac{1}{(s^2+1)(s^2-1)}\right] = L^{-1}\left(\frac{1}{2}\left(\frac{1}{(s^2-1)} - \frac{1}{(s^2+1)}\right)\right)$

$= \frac{1}{2}\left[L^{-1}\left(\frac{1}{(s^2-1)}\right) - L^{-1}\left(\frac{1}{(s^2+1)}\right)\right]$

$= \frac{1}{2}$ (sinht – sint)

Hence $L^{-1}[\frac{1}{s(s^2+1)(s^2-1)}] = \int_0^t f(u)du$

i.e. $L^{-1}[\frac{1}{s(s^2+1)(s^2-1)}] = \frac{1}{2}\int_0^t (sinhu - sinu)du$

$= \frac{1}{2}(\cosh u + \cos u)_0^t$

$= \frac{1}{2}$[(cosht+cost) – (1+1)]

$= \frac{1}{2}$[cosht+cott -2]

Convolution Theorem:

If L(f(t)) = $\bar{f}$(s) and L(g(t)) = $\bar{g}$(s) then L{f(t) * g(t)} = $\bar{f}$(s). $\bar{g}$(s) (or)

$L^{-1}\{\bar{f}(s). \bar{g}(s)\}$ = f(t) * g(t)

1. Find $L^{-1}[\frac{s^2}{(s^2+4)(s^2+9)}] = -\frac{1}{5}$ [2sin2t – 3sint3t] by using Convolution Theorem.

Sol:

Let f(s) = $\frac{s}{(s^2+4^2)}$ and g(s) = $\frac{s}{(s^2+9^2)}$

Then f(t)= cos2t and g(t) = cos3t

$\therefore L^{-1}[\frac{s^2}{(s^2+4)(s^2+9)}]$ = cos2t * cos3t

$= \int_0^t cos2u . cos3(t-u)du$

$= \frac{1}{2}\int_0^t 2cos2u . cos3(t-u)du$

$= \frac{1}{2}\int_0^t [\cos(2u+3t-3u) + cos(2u-3t+3u)]du$

$= \frac{1}{2}$

$\int_0^t \{\cos[(2-3)u+3t] + \cos[(2+3)u+3t)]\}du$

$= \frac{1}{2}[\frac{\sin\{(2-3)u+3t\}}{2-3} + \frac{\sin\{(2+3)u-3t\}}{2+3}]_0^t$

$= \frac{1}{2}[\frac{1}{-1}$(sin2t-sin3t) $+\frac{1}{5}$ (sin2t+sin3t)]

$$= \frac{1}{2}\left[\sin2t\left(\frac{1}{-1}+\frac{1}{5}\right) + \sin3t\left(\frac{1}{1}+\frac{1}{5}\right)\right]$$

$$= \frac{1}{2}\left[\sin2t\left(\frac{-4}{5}\right) + \sin3t\left(\frac{6}{5}\right)\right]$$

$$= \frac{1}{5}[-2\sin2t + 3\sin3t]$$

$$= \frac{-1}{5}[2\sin2t - 3\sin3t]$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+4)(s^2+9)}\right] = \frac{-1}{5}[2\sin2t - 3\sin3t]$$

2. Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ using the Convolution theorem.

Sol: $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)} \cdot \frac{1}{(s^2+a^2)}\right]$

Let $\bar{f}(s) = \frac{s}{(s^2+a^2)}$ and $\bar{g}(s) = \frac{1}{(s^2+a^2)}$

$L^{-1}\{\bar{f}(s)\} = L^{-1}\{\frac{s}{(s^2+a^2)}\} = \cos at = f(t)$ and

$L^{-1}\{\bar{g}(s)\} = L^{-1}\{\frac{1}{(s^2+a^2)}\} = \frac{1}{a}\sin at = g(t)$

∴ By the Convolution theorem,

$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = (\cos at) * (\frac{1}{a}\sin at)$

$$= \frac{1}{a}\int_0^t (\cos au \sin a(t-u)]du$$

$$= \frac{1}{2a}\left[\int_0^t (\sin(au + at - au) - \sin(au - at + au)\right]du$$

$$= \frac{1}{2a}\left[\int_0^t (\sin(at) - \sin(2au - at)\right]du$$

$$= \frac{1}{2a}\left[\sin at.u + \frac{1}{2a}\cos(2au - at)\right]_0^t$$

$$= \frac{1}{2a}[t\sin at + \frac{1}{2a}\cos at - \frac{1}{2a}\cos t at] = \frac{t}{2a}\sin at$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t}{2a}\sin at$$

3. Find $L^{-1}\left[\frac{s+3}{(s^2-10s+29)}\right]$

Sol: $L^{-1}\left[\frac{s+3}{(s^2-10s+29)}\right] = L^{-1}\left[\frac{s+3}{(s-5)^2+2^2)}\right]$

$$= L^{-1}\left[\frac{(s-5)+8}{(s-5)^2+2^2)}\right]$$

$= e^{5t}L^{-1}[\frac{s+8}{(s)^2+2^2}]$ since by first shifting

theorem

$= e^{5t}\{L^{-1}[\frac{s}{(s)^2+4}] + 8L^{-1}[\frac{1}{(s)^2+4}]\}$

$= e^{5t}[\cos 2t + 8.\frac{1}{2}\sin 2t]$

$= e^{5t}[\cos 2t + 4\sin 2t]$

$\therefore L^{-1}[\frac{s+3}{(s^2-10s+29)}] = e^{5t}[\cos 2t + 4\sin 2t]$

19. Find $L^{-1}[2 + s\log(\frac{s-1}{s+1})]$

Let $\bar{f}(s) = \log(\frac{s-1}{s+1}) = \log(s-1) - \log(s+1)$

$\bar{f}^1(s) = \frac{1}{s-1} - \frac{1}{s+1}$

Now $L^{-1}[\bar{f}^1(s)] = L^{-1}[\frac{1}{s-1} - \frac{1}{s+1}]$

i.e $(-1)tf(t) = (e^t - e^{-t})$

$f(t) = L^{-1}[\bar{f}(s)]$

$= \frac{-1}{t}(e^t - e^{-t})$

Thus $f(t) = L^{-1}[\log(\frac{s-1}{s+1})] = \frac{-2}{t}[\frac{(e^t-e^{-t})}{2}] = \frac{-2}{t}\sinh t$

Also $f(0) = 0$

$\therefore L^{-1}[s\log(\frac{s-1}{s+1})] = L^{-1}[s.\bar{f}(s)] = f^1(t)$

$= -2\frac{d}{dt}[\frac{\sinh t}{t}]$

$= \frac{2}{t^2}[\sinh t - t\cosh t]$

Hence $L^{-1}[2 + s\log(\frac{s-1}{s+1})] = L^{-1}(2) + L^{-1}[s\log(\frac{s-1}{s+1})]$

$= \frac{2}{t^2}[\sinh t - t\cosh t]$ since $L^{-1}(2)$ doesnot

exist.

Solution of O.D.E equations with constant co-efficient:

$L[f^n(t)] = S^nL\{f(t)] - S^{n-1}f(0) - S^{n-2}f^1(0)$.............

$f^{n-1}(0)$

20. Using Laplace transform Solve $(D^3 - D^2 + 4D - 4)\, y = 68e^x$ sin2x, y=1, Dy= -19,

$D^2y = -37$ at x = 0

Given differential equation can be written as

$y^{111} - y^{11} + 4y^1 - 4y = 68e^x \sin 2x$ -----------------(1)

taking L.T on B.S of equation (1) we get,

$L(y^{111}) - L(y^{11}) + 4L(y^1) - 4L(y) = 68L(e^x \sin 2x)$

i.e. $[s^3L(y) - s^2y(0) - sy^1(0) - y^{11}(0)] - [s^2L(y) - sy(0) - y^1(0)]$

$+ 4[s.L(y) - y(0)] - 4L(y) = 68. \frac{2}{(s-1)^2+2^2}$.....(2)

using the given conditions y(0) = 1, $y^1(0) = -19$, $y^{11}(0) = -37$

(2) reduces to

$[s^3L(y) - s^2 - 19s + 37] - [s^2L(y) - s + 19] + 4[s.L(y) - 1] - 4L(y) = \frac{136}{(s-1)^2+2^2}$

i.e. $(s^3 - s^2 + 4s - 4)\, L(y) - s^2 + 20s + 14 = \frac{136}{(s)^2 - 2s + 5}$

i.e. $(s^3 - s^2 + 4s - 4)\, L(y) = \frac{136}{(s)^2 - 2s + 5} + s^2 - 20s - 14$

$(s-1)(s^2+4)\, L(y) = \frac{s^4 - 22s^3 + 31s^2 - 72s - 70}{s^2 - 2s + 5}$

i.e. $L(y) = \frac{s^4 - 22s^3 + 31s^2 - 72s - 70}{(s-1)(s^2+4)(s^2-2s+5)}$

or $y = L^{-1}\left[\frac{s^4 - 22s^3 + 31s^2 - 72s - 70}{(s-1)(s^2+4)(s^2-2s+5)}\right]$

21. Solve the D.E $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} - 12x = e^{3t}$ given that x(0) = 1 and $x^1(0) = -2$ using Laplace transforms.

Given equation can be written as

$x^{11} - 4x^1 - 12x = e^{3t}$

taking L.T on both sides we get,

$L(x^{11}) - 4L(x^{1}) - 12L(x) = L(e^{3t})$

Using the given conditions , it reduces to

$[s^2 \cdot L(x) - s + 2] - 4[s.L(x) - 1] - 12L(x) = \frac{1}{s-3}$

i.e. $(s^2 - 4s - 12)\, L(x) = \frac{1}{s-3} + s - 6$

or $(s+2)(s-6)\, L(x) = \frac{1}{s-3} + s - 6$

or $L(x) = \frac{1}{(s+2)(s-3)(s-6)} + L^{-1}[\frac{1}{s+2}]$

i.e. $x = L^{-1}[\frac{1}{(s+2)(s-3)(s-6)}] + L^{-1}[\frac{1}{s+2}]$

Let $\frac{1}{(s+2)(s-3)(s-6)} = \frac{A}{s+2} + \frac{B}{s-3} + \frac{C}{s-6}$

$\therefore 1 = A(s-3)(s-6) + B(s+2)(s-6) + c(s+2)(s-3)$

Put s=-2 in (3)

$\therefore 1 = 40A \Rightarrow A = \frac{1}{40}$

Put s=3in (3)

$\therefore 1 = -15B \rightarrow B = -15$

Put s=6 in (3)

$\therefore 1 = 24C \Rightarrow C = \frac{1}{24}$

Substitute A, B and C we get

$$\frac{1}{(s+2)(s-3)(s-6)} = \frac{1}{40(s+2)} - \frac{1}{15(s-3)} + \frac{1}{24(s-6)}$$

Hence $x = \frac{1}{40} L^{-1}[\frac{1}{s+2}] - \frac{1}{15} L^{-1}[\frac{1}{s-3}] + \frac{1}{24} L^{-1}[\frac{1}{s-6}] + L^{-1}[\frac{1}{s+2}]$

$= \frac{1}{40} e^{-2t} - \frac{1}{15} e^{3t} + \frac{1}{24} e^{6t} + e^{-2t}$

$= \frac{41}{40} e^{-2t} - \frac{1}{15} e^{3t} + \frac{1}{24} e^{6t}$

22. Using Laplace transform, Solve $(D^2 + 4D + 5)\, y = 5$ Given that $y(0) = 0$ and $y^{11}(0) = 0$.

Sol: Given differential equation can be written as

$y^{11}+4y^{1}+5y = 5$

taking L.T on both sides we get

$L(y^{11})+4L(y^{1})+5L(y) = L(5)$

$\rightarrow[s^2L(y) - sy(0) - y^1(0)] + 4[s.L(y) - y(0) + 5L(y)] = 5.L(1)$

Using the given conditions, it reduces to

$(s^2+4s+5)\, L(y) = \frac{5}{s}$

$\therefore L(y) = \frac{5}{s(s^2+4s+5)} = \frac{1}{s} - \frac{s+4}{(s^2+4s+5)}$

$\therefore y = L^{-1}[\frac{1}{s} - \frac{s+4}{(s^2+4s+5)}]$

$= L^{-1}[\frac{1}{s}] - L^{-1}[\frac{(s+2)+2}{(s+2)^2+1)}]$

$= 1 - e^{\ 2t} . L^{-1}[\frac{(s+2)}{(s^2+1)}]$

$= 1 - e^{-2t} . L^{-1}[\frac{s}{(s^2+1)} + \frac{2}{(s^2+1)}]$

$= (1 - e^{-2t})(\cos t+2\sin t)$

23. Find $L^{-1}(\cot^{-1}s)$

Sol: $\bar{f}(s) = \cot^{-1}s$

Then $\bar{f}^1(s) = \frac{d}{ds}(\cot^{-1}s) = \frac{-1}{1+s^2}$

We have $L^{-1}(\bar{f}(s)) = (-1)\ t\ f(t)$

$\therefore f(t) = L^{-1}[\bar{f}(s)] = \frac{-1}{t} L^{-1}[\bar{f}^1(s)]$

$= \frac{-1}{t} L^{-1}[\frac{-1}{1+s^2}]$

$= \frac{-1}{t} L^{-1}[\frac{1}{1+s^2}]$

$= \frac{1}{t}\sin t$

24. If $L(f(t)) = \bar{f}(s)$ then $L(\frac{f(t)}{t}) = \int_s^\infty \bar{f}(s)ds$

Given that $L(f(t)) = \bar{f}(s)$

Then $\bar{f}(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)dt$

Integrating on both sides w.r.to 's' from s to ∞

$\therefore \int_s^\infty \bar{f}(s)ds = \int_s^\infty [\int_0^\infty e^{-st} f(t)dt]\, ds$

Interchanging the order of integration in the repeated integrals as s and t are independent variables, we have

$$\int_s^\infty f(s)ds = \int_0^\infty dt[\int_s^\infty e^{-st} f(t)ds]$$

$$= \int_0^\infty f(t)[\int_s^\infty e^{-st}\, ds]dt$$

$$= \int_0^\infty f(t)\, [\frac{e^{-st}}{-t}]_s^\infty dt$$

$$= \int_0^\infty f(t)\, [\frac{0- e^{-st}}{-t}]\, dt$$

$$= \int_0^\infty f(t)\, [\frac{e^{-st}}{t}]\, dt$$

$$= \int_0^\infty e^{-st}\, [\frac{f(t)}{t}]\, dt = \int_s^\infty \bar{f}(s)ds$$

$$\therefore L(\frac{f(t)}{t}) = \int_s^\infty \bar{f}(s)ds$$

25. Find $L[\int_0^t e^{-t} \text{costat}]$

Let $f(t) = e^{-t} \cos t$

$\therefore L(f(t)) = L(e^{-t} \cos t)$

$= (\frac{s}{(s^2+1)})$ changes s to s+1

$= \frac{s+1}{(s^2+2s+2)} = \bar{f}(s)$

Using the theorem of L.T of integrals,

$L[\int_0^t e^{-st} \text{costat}] = \frac{1}{s}\bar{f}(s)$

$= \frac{s+1}{s(s^2+2s+2)}$

26. Using Laplace transform evaluate $\int_0^\infty \frac{e^{-t} - e^{-2t}}{t} dt$

The given interval same as $\int_0^\infty e^{-st} \frac{e^{\ t} - e^{\ 2t}}{t} dt$ where s=0 -----------(1)

$$L(\frac{e^{-t} - e^{-2t}}{t}) = \int_s^\infty L(e^{-t} - e^{-2t})\, ds$$

Since $L(\frac{f(t)}{t}) = \int_s^\infty \bar{f}(s)ds$

$$= \int_s^\infty (\frac{1}{s+1} - \frac{1}{s+2})\, ds$$

$$= [\log(s+1) - \log(s+2)]_s^\infty$$

$$= [\log \{\frac{(s+1)}{(s+2)}\}]_s^\infty$$

$$= [\log \{\frac{\left(1+\frac{1}{s}\right)}{\left(1+\frac{2}{s}\right)}\}]_s^\infty$$

$$= \log 1 - \log \{\frac{\left(1+\frac{1}{s}\right)}{\left(1+\frac{2}{s}\right)}\}$$

$$= -\log \{\frac{\left(1+\frac{1}{s}\right)}{\left(1+\frac{2}{s}\right)}\}$$

i.e. $L(\frac{e^{-t} - e^{-2t}}{t}) = -\log \{\frac{\left(1+\frac{1}{s}\right)}{\left(1+\frac{2}{s}\right)}\} = -\log(\frac{s+1}{s+2})$

by using (1) we get,

$$\int_0^\infty e^{-st} (\frac{e^{-t} - e^{-2t}}{t})dt = L(\frac{e^{-t} - e^{-2t}}{t}) = -\log(\frac{s+1}{s+2})$$

Put s=0

$$= \int_0^\infty \frac{e^{-t} - e^{-2t}}{t} dt = -\log(\tfrac{1}{2}) = -\log 2$$

$$\therefore \int_0^\infty \frac{e^{-t} - e^{-2t}}{t} dt = -\log 2$$

27. Find inverse Laplace transforms $\frac{s+3}{(s^2+6s+13)^2}$

Let $\bar{f}(s) = \frac{s+3}{(s^2+6s+13)^2}$

$$= \frac{s+3}{(s+3)^2+2^2)^2}$$

$L^{-1}(f(s)) = L^{-1}[\frac{s+3}{(s+3)^2+2^2)^2}]$

$$= e^{-3t} L^{-1}[\frac{s}{(s^2+2^2)^2}]$$

$$= e^{-3t} \cdot \frac{t}{2.2} \sin 2t$$

since $L^{-1}[\frac{s}{(s^2+2^2)^2}] = \frac{t}{2.a} \sin 2t$

$= \frac{t}{4} e^{-3t} \cdot \sin 2t$

28. Evaluate $L^{-1}[\frac{1+e^{-\pi s}}{(s^2+1)}]$

Sol: $L^{-1}[\frac{e^{-\pi s}}{(s^2+1)}] = L^{-1}[\frac{1}{(s^2+1)}] + L^{-1}[\frac{e^{-\pi s}}{(s^2+1)}]$

Since $L^{-1}[\frac{1}{(s^2+1)}]$ = sint = f(t) say

By second shifting theorem

$$L^{-1}[\frac{e^{-\pi s}}{(s^2+1)}] = \begin{cases} \sin(t-\pi), & t>\pi \\ 0, & t<\pi \end{cases}$$

Or $L^{-1}[\frac{e^{-\pi s}}{(s^2+1)}]$ =sin(t - π) H(t - π)

= sintH(t - π)

Hence $L^{-1}[\frac{e^{-\pi s}}{(s^2+1)}]$ = sint – sint.H(t - π) = sint[1-H(t - π)]

Where H(t - π) is the Heaviside unit step function.

29. Solve the Differential equation $y^{11} + n^2 y$ = a sin(nt+2), y(0)= 0 and $y^1(0) = 0$ using Laplace transform. or

Using Laplace transform Solve $(D^2 + n^2)x$= a sin(nt+2), x=Dx=0 at t=0

Sol: Given equation can be written as $x^{11} + n^2 x$ = a (sinnt cos2 + cosnt sin2)

Taking Laplace Transform on both sides we get,

$L(x^{11})+n^2L(x)$ = a cos2. L(sin nt) + a sin2. L(cosnt)

$[s^2L(x) - s(x)(0) - x^1(0)] + n^2(X) = a \cos 2 (\frac{n}{s^2+n^2}) = a \sin 2(\frac{s}{s^2+n^2})$

Using the given condition

$L(x) = a\cos 2 . (\frac{n}{(s^2+n^2)^2}) + a \sin 2 .\frac{s}{(s^2+n^2)^2}$

Taking inverse Laplace Transform on both sides, we get x = ancos2. $L^{-1}\left[\frac{1}{(s^2+n^2)^2}\right]$ + a sin2. $L^{-1}\left[\frac{s}{(s^2+n^2)^2}\right]$

We have $L^{-1}\left[\frac{\bar{f}(s)}{s}\right] = \int_0^t f(t)dt$

$L^{-1}\left[\frac{1}{(s^2+n^2)^2}\right] = \frac{t}{2n}$sinnt

$\therefore L^{-1}\left[\frac{1}{(s^2+n^2)^2}\right] = L^{-1}\left[\frac{1}{s} \cdot \frac{s}{(s^2+n^2)^2}\right]$

$= \frac{1}{2n}\int_0^t t\, sinnt\, dt$

$= \frac{1}{2n^3}$[-ntcosnt + sinnt]

$\therefore$ x = an cos2. $\frac{1}{2n^3}$[-ntcosnt + sinnt] + a sin2. $\frac{t}{2n}$sinnt

$= \frac{a}{2n^2}$[-nt cos2cosnt + cos2. Sinnt +nt sin2sinnt]

$= \frac{a}{2n^2}$[sinnt cos2 – nt(cosnt. Cos2 - sinnt sin2]

$= \frac{a}{2n^2}$[sinnt cos2 – nt(cosnt+2)]

UNIT - I

Laplace Transform:

Objective type Questions:

1.$L(e^{3t})$

(a) $1/s + 1$ (b) $1/s - 3$ (c) $1/s + 3$ (d) $1/s$

2. $L(t^2)$

(a) $1/s^2$ (b) $1/s^3$ (c) $2/s^3$ (d) $2/s^4$

3. $L(t^5)$

(a) $1/s^5$ (b) $1/s^6$ (c) $24/s^5$ (d) $120/s^6$

4. $L(\sin^2 2t)$

(a) $s^2/s(s^2 + 16)$ (b) $s^2 + 2/s(s^2 + 16)$ (c) $s^2 + 4/s(s^2 + 16)$ (d) $s^2+8/s(s^2+16)$

5. $L(\cosh 2t)$

(a) $2/s^2 - 4$ (b) $s/s^2 - 4$ (c) $s/s^2 + 4$ (d) 2/s2 + 4

6. $L(te^{2t})$

(a) $1/(s - 2)^2$ (b) $s/(s - 2)^2$ (c) $s + 2/(s - 2)^2$ (d) $s + 2/s^2 + 4$

7. $L(t\sinh t)$=

(a) $2/(s^2 - 1)^2$ (b) $s/(s^2 - 1)^2$ (c) $2s/(s^2 - 1)^2$ (d) $s - 2/(s^2 - 1)^2$

8. $L^{-1}[1/s - 5]$

(a) e^{5t} (b) e^{-5t} (c) sint (d) te^{5t}

9. $L^{-1}[2/s - 9)$

(a) $2e^{-9t}$ (b) 2sint (c) e^{9t} (d) e^{5t}

10. $L^{-1}[6/s^4]$

(a) t^2 (b) t^3 (c) t^4 (d) $t^3/6$

11. $L^{-1}[1/(s + 2)(s - 4)]$

(a) $e^{-2t} - e^{-4t}$ (b) $e^{4t} - e^{-2t}$ (c) $1/6[e^{-4t} - e^{-2t}]$ (d) $½[e^{4t} - e^{-2t}]$

12. L(sintcost)

(a) $2/s^2 + 4$ (b) $1/s^2 + 4$ (c) $2/s^2 - 4$ (d) $s/s^2 + 4$

13. $L^{-1}[s + 5/s^2 - 4s + 5]$

(a) $e^{-2t}\cos 2t$ (b) $e^{-2t}\sin t$ (c) $e^{-2t}\cos t + 5e^{-2t}\sin t$ (d) $e^{-2t}(\sin t+\cos t)$

14. $L[e^{2t} - e^{3t}/t]$

(a) log[(s - 3)/s -2] (b) log [s - 2/s - 3] (c) log[s + 4/s – 9] (d) log[s + 2/s + 3]

15. $L(\cos^2 t)$

(a) $s/s^2 + 4$ (b) $s/2(s^2 + 4)$ (c) $1/2s + ½(s^2 + 4)$ (d) $1/2s + s/2(s^2 + 4)$

16. $L^{-1}[\log s + 6/s – 2]$

(a) $e^{2t} - e^{-6t}/t$ (b) $e^{-6t} - e^{2t}/t$ (c) $e^{2t} + e^{6t}/t$ (d) $e^{2t} + e^{-6t}/t$

17. $L^{-1}[5/s^5]$

(a) $5t^4$ (b) $t^4/24$ (c) $(5/24)\, t^4$ (d) t^5

18. $L^{-1}[3s/s^2 + 16]$

(a) cos4t/3 (b) 3co4t (c) cos4t (d) 3sin4t

19.If [F(t-a)]=0 , 0<t<a then L[F(t-a)]=

(a) e^{-as} (b) s e^{-as} (c) $\frac{e^{\;as}}{s}$ (d) $\frac{e^{\;as}}{s^2}$

20. L(sinh4t)

(a) $\frac{4}{s^2-16}$ (b) $\frac{s}{s^2-16}$ (c) $\frac{s}{s^2+16}$ (d) $\frac{4}{s^2+16}$

21. If L(f(t))=$\frac{6}{s^2+4}$ then L$\left[\int_0^1 f(t)dt\right]$ =

(a) $\frac{6s}{s^2+4}$ (b) $\frac{s}{s^2+4}$ (c) $\frac{6}{s^2+4}$ (d) $\frac{6}{s(s^2+4)}$

22. L^{-1} $[1/s^n]$ is possible only when n is

(a) Positive integer (b) zero (c) negative Integer (d) negative rational

23. L^{-1} $\left[\frac{e^{-\pi s}}{(s^2+1)}\right]$ =

(a) cost u(t-π) (b) sint u(t-π) (c) -sint u(t-π) (d) -cost u(t-π)

24. $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$

(a) $\frac{1}{(b-a)}$ $(e^{-at} - e^{-bt})$ (b) $\frac{1}{(a-b)}(e^{-bt} - e^{at})$

(c) $\frac{1}{(a+b)}(e^{-at} + e^{-bt})$ (d) $\frac{1}{(a-b)}(e^{at} + e^{bt})$

25. L^{-1} [1]

(a) 0 (b)1 (c)δt (d) δ(t-1)

26. L^{-1} $[2^t]$

(a) $\frac{1}{s-log2}$ (b) $\frac{1}{s}$ (c) $\frac{1}{S+log2}$ (d) $\frac{1}{log2}$

27. L^{-1} $[3!/s^4]$

(a) t^3 (b) t^5 (c) $\frac{t^3}{logt}$ (d) t^8

28. Laplace transform of f(t) is defined as

(a) $\int_0^\infty e^{-st} f(t)dt$ (b) $\int_0^\infty f(t)dt$ (c) $\int_0^\infty e^{st} f(t)dt$ (d) $\int_0^\infty e^{st} dt$

29. Γ(n) =

(a) $\int_0^\infty e^{-x}x^{n-1}dx$ (b) $\int_0^\infty e^x x^{n+1}dx$ (c) $\int_0^\infty e^x dx$ (d) none

30. $\Gamma\left(\frac{1}{2}\right)$

(a) $\sqrt{\pi}$ (b) π (c) π^2 (d) 0

31. When s>a $L(e^{at}\ t^n)$

(a) $\frac{n!}{(s\ \ a)^n}$ (b) $\frac{n}{(s-a)^n}$ (c) $\frac{n!}{(s+a)^n}$ (d) $\frac{n}{s+a}$

32. $\int_0^\infty \frac{sint}{t}\,dt$

(a) 0 (b) 1 (c) $\frac{\pi}{2}$ (d) ∞

33. If Lt(t-a) is a unit step function,L{H(t-a)}

(a) $\frac{e^{-as}}{s}$ (b) $\frac{1}{s}$ (c) e^a (d) $\frac{e^{as}}{s}$

34. $L(\sqrt{t}\ )$

(a) $\frac{\sqrt{t}}{2s^{\frac{3}{2}}}$ (b) $\frac{\pi}{2s}$ (c) $\frac{1}{s^{\frac{3}{2}}}$ (d) $\frac{\pi}{2s^{\frac{3}{2}}}$

35. When $|s|$ >k ,L(sinhkt)

(a) $\frac{k}{s^2-k^2}$ (b) $\frac{k}{s^2}$ (c) $\frac{k}{s^2+k^2}$ (d) $\frac{1}{s^2+k^2}$

36. $\Gamma(n+1)$

(a) n! (b) n (c)n-1 (d)$\frac{n+1}{2}$

37. The value of $\int_0^\infty e^{-2t}cos3tdt$ ________________

38. $L^{-1}\left[\frac{1}{(4s^2-25)}\right]=$ ____________________

39. $L^{-1}\left[\frac{1}{\sqrt{s+4}}\right]=$ ____________________

40. L $\left[\frac{1-e^t}{t}\right]=$ ___________________________

41. $L^{-1}\left[\frac{1}{s(s^2+w^2)}\right]=$ ____________________

42. $L^{-1}\left[\frac{1}{\sqrt{s}}\right]=$ ____________________

43. $L^{-1}\left[\frac{1}{s^{\frac{5}{2}}}\right]=$ ____________________

44. $L[t^2e^t]=$ ____________________

45. If L[f(t)]=$\frac{1}{s}\ e^{\frac{-1}{s}}$ then L[f(2t)]=____________________

46. If f(0)=0 then $L(f^1\ (t))$ = ____________________

47. $L^{-1}\left[\frac{1}{(2s-5)}\right]=$ ____________________

48. $L^{-1}\left[\frac{s}{(s^2-a^2)^2}\right]=$ ____________________

49. L $[f^1\ (t)]=$ ____________________

50. $L^{-1}\left[\frac{1}{(s-a)(s-b)}\right]=$ ____________________

UNIT-II

Beta and gamma functions

1. THE GAMMA FUNCTION

The gamma function may be regarded as a generalization of n! (n-factorial), where n is any positive integer to x!, where x is any real number. (With limited exceptions, the discussion that follows will be restricted to positive real numbers.) Such an extension does not seem reasonable, yet, in certain ways, the gamma function defined by the improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \qquad (1)$$

meets the challenge. This integral has proved valuable in applications. However, because it cannot be r presented through elementary functions, establishment of its properties take some effort. Some of the important ones are outlined below.

The gamma function is convergent for x > 0. It follows from eq.(1) that

From (1): $\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt$

Integrating by parts

$$\Gamma(x+1) = \left[t^x\left(\frac{e^{-t}}{-1}\right)\right]_0^\infty + x\int_0^\infty e^{-t}t^{x-1}\,dt$$

$$= \{0-0\} + x\Gamma(x)$$

$$\therefore\ \Gamma(x+1) = x\Gamma(x) \qquad (2)$$

This is a fundamental recurrence relation for gamma functions. It can also be written as

$$\Gamma(x) = (x - 1)\Gamma(x - 1).$$

A number of other results can be derived from this as follows: If $x = n$, a positive integer, i.e. if $n \geq 1$, then

$$\begin{aligned}
\Gamma(n+1) &= n\Gamma(n). \\
&= n(n-1)\Gamma(n-1) \qquad \text{since } \Gamma(n) = (n-1)\Gamma(n-1) \\
&= n(n-1)(n-2)\Gamma(n-2) \qquad \text{since } \Gamma(n-1) = (n-2)\Gamma(n-2) \\
&= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= n(n-1)(n-2)(n-3)\ldots 1\Gamma(1) \\
&= n!\Gamma(1)
\end{aligned}$$

But $\Gamma(1) = \int_0^\infty t^0 e^{-t} dt = [-e^{-t}]_0^\infty = 1$

$$\Rightarrow \quad \Gamma(n+1) = n! \qquad (3)$$

Example:

$\Gamma(7) = 6! = 720, \quad \Gamma(8) = 7! = 5040, \quad \Gamma(9) = 40320$

We can also use the recurrence relation in reverse

$\Gamma(x+1) = x\Gamma(x) \quad \Rightarrow \quad \Gamma(x) = \frac{\Gamma(x+1)}{x}$

What happens when $x = \frac{1}{2}$? We will investigate.

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty t^{-1/2} e^{-t} \, dt$$

Putting $t = u^2$, $dt = 2u\,du$, then

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty u^{-1} e^{-u^2} 2u \, du = 2\int_0^\infty e^{-u^2} \, du.$$

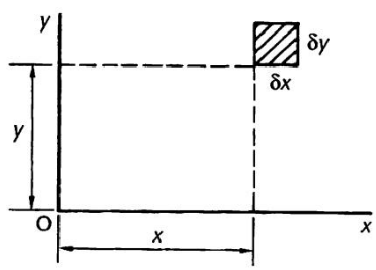
Unfortunately, $\int_0^\infty e^{-u^2} \, du$ cannot easily be determined by normal means. It is, however, important, so we have to find a way of getting round the difficulty.

Evaluation of $\int_0^\infty e^{-x^2} dx$

Let $I = \int_0^\infty e^{-x^2} dx$, then also $I = \int_0^\infty e^{-y^2} dy$

$$\therefore \; I^2 = \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dx \, dy$$

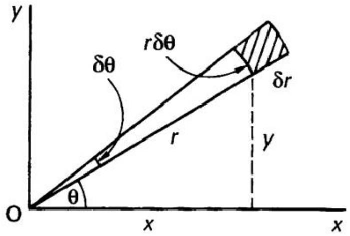
$\delta a = \delta x \, \delta y$ represents an element of area in the x–y plane and the integration with the stated limits covers the whole of the first quadrant.



Converting to polar coordinates, the element of area $\delta a = r\,\delta\theta\,\delta r$. Also, $x^2 + y^2 = r^2$

$$\therefore \; e^{-(x^2+y^2)} = e^{-r^2}$$

For the integration to cover the same region as before,



the limits of r are $r = 0$ to $r = \infty$

the limits of θ are $\theta = 0$ to $\theta = \pi/2$.

$$\therefore \; I^2 = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r \, dr \, d\theta = \int_0^{\pi/2} \left[-\frac{e^{-r^2}}{2}\right]_0^\infty d\theta$$

$$= \int_0^{\pi/2} \left(\frac{1}{2}\right) d\theta = \left[\frac{\theta}{2}\right]_0^{\pi/2} = \frac{\pi}{4}$$

Before that diversion, we had established that

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-u^2}\,du$$

We now know that $\int_0^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$ $\quad\therefore\ \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

From this, using the recurrence relation $\Gamma(x+1) = x\Gamma(x)$, we can obtain the following

$$\Gamma(\tfrac{3}{2}) = \tfrac{1}{2}\ \Gamma(\tfrac{1}{2}) = \tfrac{1}{2}(\sqrt{\pi}) \qquad \therefore\ \Gamma(\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2}$$

$$\Gamma(\tfrac{5}{2}) = \tfrac{3}{2}\ \Gamma(\tfrac{3}{2}) = \tfrac{3}{2}\left(\frac{\sqrt{\pi}}{2}\right) \qquad \therefore\ \Gamma(\tfrac{5}{2}) = \frac{3\sqrt{\pi}}{4}$$

Negative values of x

Since $\Gamma(x) = \frac{\Gamma(x+1)}{x}$, then as $x \to 0$, $\Gamma(x) \to \infty$ $\therefore$ $\Gamma(0) = \infty$.

The same result occurs for all negative integral values of x – which does not follow from the original definition, but which is obtainable from the recurrence relation.

Because at $x = -1$, $\quad \Gamma(-1) = \frac{\Gamma(0)}{-1} = \infty$

$x = -2$, $\quad \Gamma(-2) = \frac{\Gamma(-1)}{-2} = \infty$ etc.

Also, at $x = -\tfrac{1}{2}$, $\quad \Gamma(-\tfrac{1}{2}) = \frac{\Gamma(\frac{1}{2})}{-\frac{1}{2}} = -2\sqrt{\pi}$

and at $x = -\tfrac{3}{2}$, $\quad \Gamma(-\tfrac{3}{2}) = \frac{\Gamma(-\frac{1}{2})}{-\frac{3}{2}} = \frac{4}{3}\sqrt{\pi}$

So we have

(a) For n a positive integer

$\Gamma(n+1) = n\Gamma(n) = n!$

$\Gamma(1) = 1;\quad \Gamma(0) = \infty;\quad \Gamma(-n) = \pm\infty$

(b) $\Gamma(\tfrac{1}{2}) = \sqrt{\pi};\qquad \Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi}$

$\Gamma(\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2};\qquad \Gamma(-\tfrac{3}{2}) = \frac{4}{3}\sqrt{\pi}$

$\Gamma(\tfrac{5}{2}) = \frac{3\sqrt{\pi}}{4};\qquad \Gamma(-\tfrac{5}{2}) = -\frac{8}{15}\sqrt{\pi}$

$\Gamma(\tfrac{7}{2}) = \frac{15\sqrt{\pi}}{8};\qquad \Gamma(-\tfrac{7}{2}) = \frac{16}{105}\sqrt{\pi}$

Example:

Evaluate $\int_0^\infty x^7 e^{-x}\,dx$.

We recognise this as the standard form of the gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt \qquad \text{with the variables changed.}$$

It is often convenient to write the gamma function as

$$\Gamma(v) = \int_0^\infty x^{v-1}e^{-x}\,dx$$

Our example then becomes

$$I = \int_0^\infty x^7 e^{-x}\,dx = \int_0^\infty x^{v-1}e^{-x}\,dx \qquad \text{where } v = \ldots\ldots\ldots\ldots$$

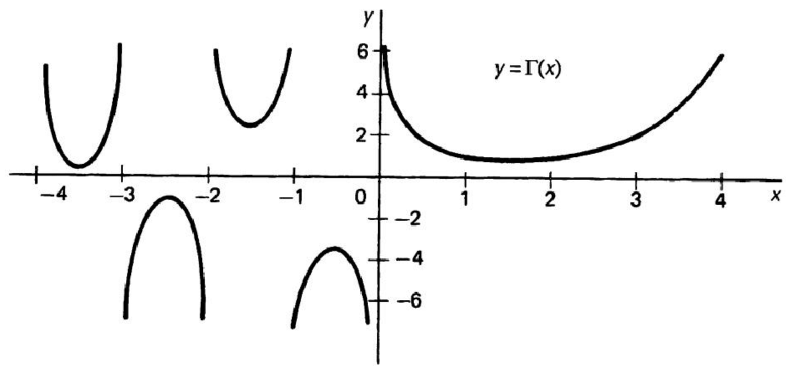
$$\text{i.e. } \int_0^\infty x^7 e^{-x}\,dx = \Gamma(8) = 7! = 5040$$

Graph of $y = \Gamma(x)$

Values of $\Gamma(x)$ for a range of positive values of x are available in tabulated form in various sets of mathematical tables. These, together with the results established above, enable us to draw the graph of $y = \Gamma(x)$.

x	0	0·5	1·0	1·5	2·0	2·5	3·0	3·5	4·0
$\Gamma(x)$	∞	1·772	1·000	0·886	1·000	1·329	2·000	3·323	6·000

x	−0·5	−1·5	−2·5	−3·5
$\Gamma(x)$	−3·545	2·363	−0·945	0·270



Example:

Evaluate $\int_0^\infty x^3 e^{-4x}\,dx$.

If we compare this with $\Gamma(v) = \int_0^\infty x^{v-1} e^{-x}\,dx$, we must reduce the power of e to a single variable, i.e. put $y = 4x$, and we use this substitution to convert the whole integral into the required form.

$y = 4x \quad \therefore\ dy = 4\,dx \qquad$ Limits remain unchanged.

The integral now becomes ....

$$I = \int_0^\infty \left(\frac{y}{4}\right)^3 e^{-y}\,\frac{dy}{4}$$

$$\therefore\ I = \frac{1}{4^4}\int_0^\infty y^3 e^{-y}\,dy = \frac{1}{4^4}\Gamma(v) \qquad \text{where } v = \ldots\ldots\ldots\ldots$$

$$\int_0^\infty y^{v-1}e^{-y}dy = \int_0^\infty y^3 e^{-y}dy \qquad \therefore\ v = 4$$

$$\therefore\ I = \frac{1}{4^4}\,\Gamma(4) = \ldots\ldots\ldots\ldots$$

$$I = \frac{1}{256}\Gamma(4) = \frac{1}{256}(3!) = \frac{6}{256} = \frac{3}{128}$$

2. THE BETA FUNCTION

The beta function is a two-parameter composition of gamma functions that has been useful enough in application to gain its own name.

The beta function $B(m, n)$, is defined by

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad (1)$$

which converges for $m > 0$ and $n > 0$.

Putting $(1-x) = u \quad \therefore \; x = 1-u \quad \therefore \; dx = -\,du$

Limits: when $x = 0,\ u = 1;\quad$ when $x = 1,\ u = 0$

$$\therefore\ B(m, n) = -\int_1^0 (1-u)^{m-1}\,u^{n-1}\,du = \int_0^1 (1-u)^{m-1}\,u^{n-1}du$$

$$= \int_0^1 u^{n-1}(1-u)^{m-1}du = B(n, m)$$

$$\therefore\ B(m, n) = B(n, m) \qquad (2)$$

Alternative form of the beta function

We had

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

If we put $x = \sin^2\theta$, the result then becomes ....

Because if $x = \sin^2\theta,\quad dx = 2\sin\theta\,\cos\theta\,d\theta.$

When $x = 0,\ \theta = 0$; when $x = 1,\ \theta = \pi/2.\ 1 - x = 1 - \sin^2\theta = \cos^2\theta$

$$\therefore\ B(m, n) = 2\int_0^{\pi/2} \sin^{2m-2}\theta\cos^{2n-2}\theta\sin\theta\,\cos\theta\,d\theta$$

$$\therefore\ B(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta \qquad (3)$$

Relation between the gamma and Beta Functions

If m and n are positive integers

$$B(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

Also, we have previously established that, for n a positive integer,

$$n! = \Gamma(n+1)$$

$\therefore\ (m-1)! = \Gamma(m)\quad$ and $\quad (n-1)! = \Gamma(n)$

and also $\ (m+n-1)! = \Gamma(m+n)$

$$\therefore\ B(m, n) = \frac{(m-1)!(n-1)!}{(m+n-1)!} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

The relation $B(m, n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ holds good even when m and n are not necessarily integers.

Application of gamma and beta functions

The use of gamma and beta functions in the evaluation of definite integrals depends largely on the ability to change the variables to Express the integral in the basic form of the beta function

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx$$

or its trigonometrical form $2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$.

Example:

Evaluate $I = \int_0^1 x^5(1-x)^4\,dx$.

Compare this with $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Then $m-1=5 \quad \therefore\ m=6$ and $n-1=4 \quad \therefore\ n=5$

$$\therefore\ I = B(6, 5) = \ldots\ldots\ldots\ldots$$

$$\boxed{I = B(6, 5) = \frac{5!\,4!}{10!} = \frac{1}{1260}}$$

Example:

Evaluate $I = \int_0^1 x^4\sqrt{1-x^2}dx$.

Comparing this with $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

we see that we have x^2 in the root, instead of a single x.

Therefore, put $x^2 = y \quad \therefore\ x = y^{\frac{1}{2}} \quad dx = \frac{1}{2}y^{-\frac{1}{2}}dy$

The limits remain unchanged. $\therefore\ I = \ldots\ldots\ldots\ldots$

$$\boxed{I = \tfrac{1}{2}B(\tfrac{5}{2}, \tfrac{3}{2})}$$

Because

$$I = \int_0^1 y^2(1-y)^{\frac{1}{2}}\frac{1}{2}y^{-\frac{1}{2}}dy = \frac{1}{2}\int_0^1 y^{\frac{3}{2}}(1-y)^{\frac{1}{2}}dy$$

$m-1=\frac{3}{2} \quad \therefore\ m=\frac{5}{2}$ and $n-1=\frac{1}{2} \quad \therefore\ n=\frac{3}{2}$

$\therefore\ I = \frac{1}{2}B(\frac{5}{2}, \frac{3}{2})$

Expressing this in gamma functions

$$\boxed{I = \frac{1}{2}\,\frac{\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{\Gamma(4)}}$$

From our previous work on gamma functions

$$\Gamma(\tfrac{3}{2}) = \frac{\sqrt{\pi}}{2};\quad \Gamma(\tfrac{5}{2}) = \frac{3\sqrt{\pi}}{4};\quad \Gamma(4) = 3!$$

$$I = \frac{1}{2}\cdot\frac{(3\sqrt{\pi}/4)(\sqrt{\pi}/2)}{3!} = \frac{\pi}{32}.$$

Now you can work through this one in much the same way. There are no tricks.

Exercises

1. Evaluate

3. Evaluate $I = \int_0^3 \frac{x^3 dx}{\sqrt{3-x}}$ (F.E.M. 170)

2. Evaluate

(a) $\frac{\Gamma(6)}{3\Gamma(4)}$ (b) $\frac{\Gamma(1{\cdot}5)}{\Gamma(2{\cdot}5)}$ (c) $\frac{\Gamma(-\frac{1}{2})}{\Gamma(\frac{1}{2})}$

(d) $\int_0^\infty x^5\, e^{-x}\, dx$ (e) $\int_0^\infty x^6\, e^{-4x^2}\, dx.$

3. Determine

(a) $\int_0^1 x^5(2-x)^4 dx$

(b) $\int_0^{\pi/2} \sin^7\theta \cos^3\theta\, d\theta$

(c) $\int_0^{\pi/8} \sin^2 4\theta \cos^5 4\theta\, d\theta.$

4. Evaluate

(a) $\frac{\Gamma(5)}{2\Gamma(3)}$; (b) $\frac{\Gamma(\frac{1}{2})}{\Gamma(-\frac{1}{2})}$; (c) $\frac{\Gamma(2{\cdot}5)}{\Gamma(3{\cdot}5)}$;

(d) $\int_0^\infty x^4 e^{-x}\, dx$; (e) $\int_0^\infty x^8 e^{-2x}\, dx.$

5. Determine

(a) $\int_0^\infty x^3 e^{-x}\, dx$; (b) $\int_0^\infty x^4 e^{-3x}\, dx$;

(c) $\int_0^\infty x^2 e^{-2x^2} dx$; (d) $\int_0^\infty \sqrt{x} \cdot e^{-\sqrt{x}}\, dx.$

6.

If m and n are positive constants, show that $\int_0^\infty x^m e^{-ax^n} dx$ can be expressed in the form $\frac{1}{n \cdot a^{(m+1)/n}} \Gamma\left(\frac{m+1}{n}\right).$

7. Evaluate the following integrals

(a) $\int_0^{1/2} x^4(1-2x)^3 dx$

(b) $\int_0^{1/\sqrt{2}} x^2\sqrt{1-2x^2}\, dx$

(c) $\int_0^{\pi/2} \sin^5\theta \cos^4\theta\, d\theta$

(d) $\int_0^{\pi/2} \sin\theta\sqrt{\cos^5\theta}\, d\theta$

(e) $\int_0^{\pi/4} \sin^3 2\theta \cos^6 2\theta\, d\theta$

(f) $\int_0^{1/3} x^2\sqrt{1-9x^2}\, dx.$

UNIT-III

Multiple Integrals& its applications

Definite Integrals: Let y =f(x) be a function of one variable define and bounded on [a,b] consider the sum $\sum_{i=1}^{n} f(x_i)\ \delta x_i$ of this sum tends to a finite limit as n =>∞ such that length of δx_i tends to O for arbitrary choice of the t_i's. The limit is define to be the definite integral $\int_a^b f(x)dx$.

The generalization of this definition to two dimensions is called a double integral and to three dimensions is called a triple integral.

Double Integral: An expression of the form $\int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)dy\ dx$ or $\int_a^b \int_{x_1(y)}^{x_2(y)} f(x,y)dx\ dy$ is called an iterated integral or double integral.

1) Evaluate $\int_0^1 \int_0^x e^{x+y}$dy dx

Ans: = $\frac{1}{2}(e-1)^2$

2) Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)$ dxdy

$$= \int_{x=0}^{1} dx \int_{y=x}^{\sqrt{x}} (x^2+y^2)dy$$

$$= \int_{x=0}^{1} dx\ [x^2 y + \frac{y^3}{3}]_x^{\sqrt{x}} dx$$

$$= \int_{x=0}^{1} [x^2\sqrt{x} + \frac{(\sqrt{x})^3}{3}] - [x^3 + \frac{x^3}{3}]]\ dx$$

$$= \int_{x=0}^{1} [x^{\frac{5}{2}} + \frac{(x)^{\frac{3}{2}}}{3} - \frac{4x^3}{3}]\ dx$$

$$= [\frac{(x)^{\frac{7}{2}}}{\frac{7}{2}} + \frac{(x)^{\frac{5}{2}}}{3.\frac{5}{2}} - \frac{4x^4}{3\ .4}]_0^1$$

$$= \frac{2}{7} + \frac{2}{15} - \frac{1}{3} = \frac{30+14-35}{105} = \frac{9}{105} = \frac{3}{35}$$

P.T $\int_1^2 \int_3^4 (xy + e^y)$ dx dy

$= \int_3^4 \int_1^2 (xy + e^y)$ dy dx

L.H.S $= \int_1^2 [\int_3^4 (xy + e^y)$ dx]dy

$= \int_1^2 [y . \frac{x^2}{2} + e^y . x]_3^4$dy

$= \int_1^2 \{[y . \frac{16}{2} + 4e^y] - [y . \frac{9}{2} + 3e^y]\}$dy

$= \int_1^2 [y . \frac{7}{2} + e^y]$dy

$= [y . \frac{7}{2} + e^y]_1^2$

$= (\frac{7}{4} * 4 + e^2) - (\frac{7}{4} + e)$

$= 7 - \frac{7}{4} + e^2 - e$

$= \frac{21}{4} + e^2 - e$

R.H.S $= \int_3^4 [\int_1^2 (xy + e^y)$ dx] dy

$= \int_3^4 [x . \frac{y^2}{2} + e^y]_1^2$dx

$= \int_3^4 \{[x . \frac{4}{2} + e^2] - [\frac{x}{2} + e]\}$dx

$= \int_3^4 [\frac{3x}{2} + e^2 - e]$ dx

$= [\frac{3}{2}\frac{x^2}{2} + e^2x - ex]_3^4$

$= (\frac{3}{4} * 9 + 3e^2 - 3e) - (\frac{3}{4} * 16 + 4e^2 - 4e)$

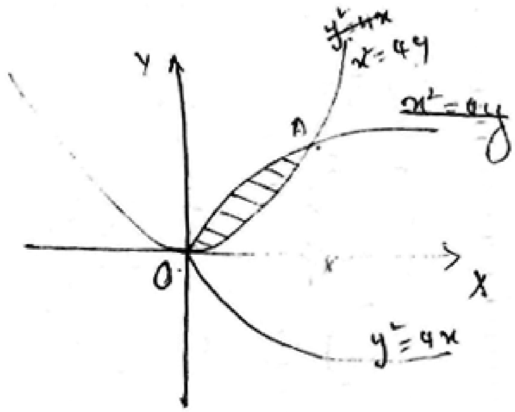
$= \frac{21}{4} + e^2 - e$

∴ L.H.S =R.H.S

3). Evaluate $\int\int y\, dx\, dy$ where R is the region bounded by the parabolas y^2 = 4x and x^2 = 4y

The co-ordinates of points O & A are

(0,0) and (4,4)



$$\int\int y\, dx\, dy = \int_{x=0}^{4} dx \int_{y=\frac{x^2}{4}}^{y=2\sqrt{x}} y\, dy$$

$$= \int_{x=0}^{4} dx \left(\frac{y^2}{4}\right)_{\frac{x^2}{4}}^{2\sqrt{x}}$$

$$= \int_{x=0}^{4} \left[\frac{4x}{2} - \frac{x^4}{2.16}\right] dx$$

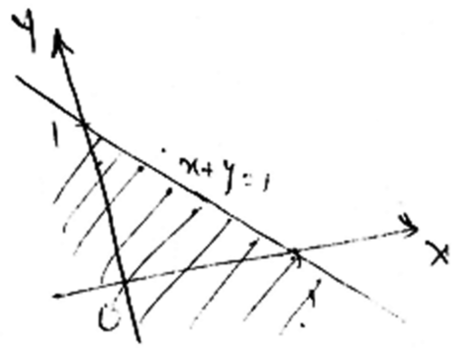
$$= \frac{1}{2}\left[4 .\frac{x^2}{2} - \frac{x^2}{16\,.\,5}\right]_0^4$$

$$= \frac{1}{2}\left[32 - \frac{64}{16}\right]$$

$$= \frac{1}{2}\left[\frac{160-64}{5}\right] = \frac{1}{2}\left[\frac{96}{5}\right] = \frac{48}{5}$$

$$\therefore \int\int y\, dx\, dy = \frac{48}{5}$$

4). Evaluate $\int\int x^2 + y^2$ dx dy in positive quadrant for which x+y≤1.



$\int\int x^2 + y^2 dxdy =$

$$= \int_{x=0}^{1} dx \int_{y=0}^{1-x} (x^2 + y^2)\, dy$$

$$= \int_{x=0}^{1} (x^2 y + \frac{y^3}{3})_0^{1-x}\, dx$$

$$= \int_0^1 [x^2(1-x) + \frac{(1-x)^3}{3}]\, dx$$

$$= \int_0^1 [x^2 - x^3 + \frac{1}{3}(1 - 3x + 3x^2 - x^3)]\, dx$$

$$= (\frac{2x^3}{3} - \frac{4}{3} \cdot \frac{x^4}{4} + \frac{1}{3}x - \frac{x^2}{2})_0^1$$

$$= \frac{2}{3} - \frac{1}{3} + \frac{1}{3} - \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6}$$

$$\therefore \int\int x^2 + y^2 dxdy = \frac{1}{6}$$

Change of order of integration:

5). Evaluate the following integrals by changing the order of integration.

Sol:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\, dx\, dy$$

The area of integration lies between y=0 which is x-axis and

$y = \sqrt{1 - x^2} \Rightarrow x^2 + y^2 = 1$

Which is a circle. Also limits of x are 0 to 1.

Hence the region of integration is OAB and is divided into vertical strip for changing the order of integration; we shall divide the region of integration into horizontal strips.

The new limits of integration become x = 0 to x= $\sqrt{1-y^2}$ and those for 'y' will be y=0 to y = 1.

Hence

$$\int_0^1\int_0^{\sqrt{1-x^2}} y^2\,dx\,dy = \int_{y=0}^1 dy\int_{x=0}^{\sqrt{1-y^2}} y^2\,dx$$

$$= \int_{y=0}^1 y^2[\,(x\,)_0^{\sqrt{1-y^2}}\,]dy$$

$$= \int_{y=0}^1 y^2\sqrt{1-y^2}dy$$

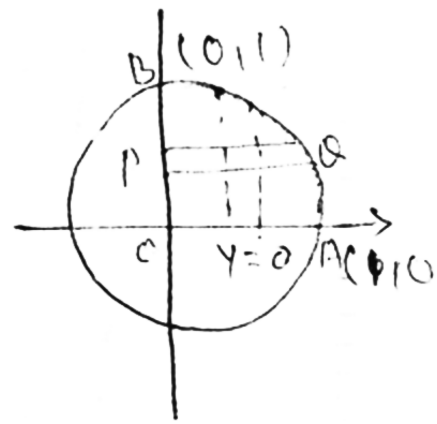
Put y = sinϑ dy=cos$\theta d\theta$

y=0 =>$\theta = 0$,

y=1 =>$\theta = \frac{\pi}{2}$

hence I= $\int_{\theta=0}^{\frac{\pi}{2}} sin^2\theta cos^2\theta d\theta$

$= \frac{1}{4}.\frac{\pi}{2}.\frac{1}{2} \quad = \frac{\pi}{16}$



6). $\int_0^a\int_y^a \frac{x}{x^2+y^2}$dy dx

Sol. The region of integration lies between x=y a straight line and passing through the origin x=a and y=0. Also the limits for y are 0 to a, which is ΔOAB and the region is divided by horizontal strips.

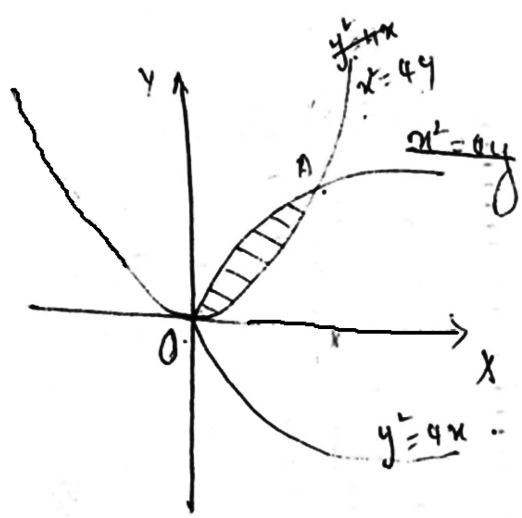
By changing the order of integration take a vertical strip PQ so that the new limits become y=0 to y=x and x varies from 0 to a.

Hence I = $\int_0^a\int_y^a \frac{x}{x^2+y^2}$dy dx

$= \int_{x=0}^a dx\int_{y=0}^x \frac{x}{x^2+y^2}$dy

=

$\int_{x=0}^a x.\ (\frac{1}{x}tan^{-1}\frac{y}{x})_0^x dx$



$$= \int_{x=0}^{a} tan^{-1}(1)\, dx$$

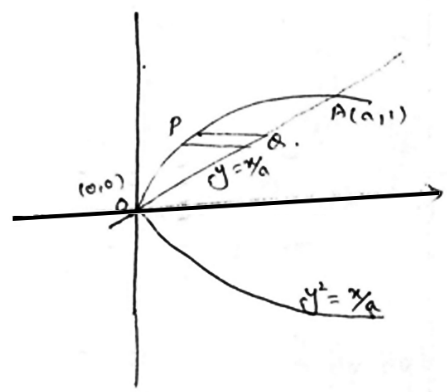
$$= \frac{\pi}{4}(x)_0^a = \frac{\pi a}{4}$$

$$\therefore \int_0^a \int_y^a \frac{x}{x^2+y^2} dy\, dx = \frac{\pi a}{4}$$

7). $\int_0^a \int_{\frac{x}{a}}^{\sqrt{\frac{x}{a}}} (x^2 + y^2)\, dx\, dy$

Sol. The region of integration is defined by y= $\sqrt{\frac{x}{a}}$ => $y^2 = \frac{x}{a}$ which is a parabola and y= $\frac{x}{a}$ => x=ay is a straight line passing through the origin. The points of intersection are O(0,0) and A(a,1). The limits for x are 0 to a.

Integration is done by taking strip parallel to y-axis. By changing the order of integration take a strip PQ parallel to x-axis. The limits for x in this case will be x=a y^2 to x=ay and that for y will be y=0 to y=1.



$$\therefore I = \int_0^a \int_{y=\frac{x}{a}}^{y=\sqrt{\frac{x}{a}}} (x^2 + y^2)\, dx\, dy$$

$$= \int_{y=0}^{1} \int_{x=ay^2}^{x=ay} (x^2 + y^2)\, dx\, dy$$

$$= \int_{y=0}^{1} \left(\frac{x^3}{3} + y^2 x\right)_{ay^2}^{ay} dy$$

$$= \int_0^1 \left(\frac{a^3y^3}{3} + ay^3 - \frac{a^3y^6}{3} - ay^4\right) dy$$

$$= \left(\frac{a^3}{3}\frac{y^4}{4} + a\frac{y^4}{4} - \frac{a^3}{3}\frac{y^7}{7} - a\frac{y^5}{5}\right)_0^1$$

$$= \frac{a^3}{12} + \frac{a}{4} - \frac{a^3}{21} - \frac{a}{5} \qquad = \frac{a^3}{28} + \frac{a}{20}$$

$$\therefore \int_0^a \int_{y=\frac{x}{a}}^{y=\sqrt{\frac{x}{a}}} (x^2 + y^2)\, dx\, dy = \frac{a^3}{28} + \frac{a}{20}$$

Change of variables:

Let x and y be functions of u and v and let x = $\emptyset(u, v)$ and y = x(u,v) then $\int_R \int f(x,y)\, dx\, dy$ is transformed into $\int_{R^1} \int f\{\emptyset(u,v), x(u,v)\}\, |\mathcal{J}|\, du\, dv$

Where $\mathcal{J} = \frac{\partial(x,y)}{\partial(u,v)}$ is the jacobian of transformation from (x,y) to(u,v) co-ordinates and R[1]is the region in the uv plane corresponding to R in the xy plane.

In polar co-ordinates x=rcosθ , y= rsinθ

$$\mathcal{J} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = r$$

$$\therefore \int_R \int f(x,y)dx\,dy = \int_{R1} \int f\{\mathrm{rcos}\theta, \mathrm{rsin}\theta\}\, r d\theta$$

8). Evaluate the following integrals by changing to polar co-ordinates.

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy$$

Since both x and y vary from 0 to ∞, the region of integration is the xoy plane, change to polar co-ordinates, x= $\mathrm{rcos}\theta$, y = $\mathrm{rsin}\theta$ dx dy = r dr dθ and $(x^2 + y^2) = r^2$. In the region of integration 'r' varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$.

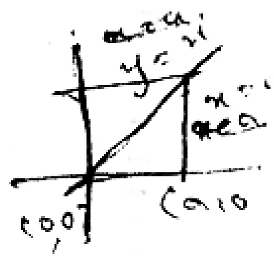
$$\therefore \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} e^{-r^2}\, r\, dr\, d\theta$$

Put t =r^2

$\therefore$dt = 2r dr

r =0 => t=0

r=∞ => t = ∞



$$I = \int_{\theta=0}^{\frac{\pi}{2}} \int_0^\infty e^{-t} \frac{dt}{2} d\theta$$

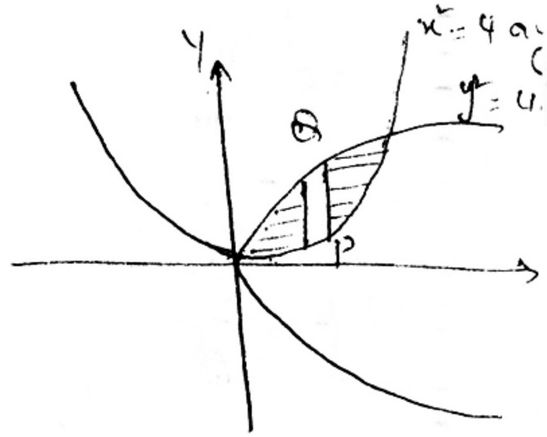
$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} [-e^{-t}]_0^\infty d\theta$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{2}} (0-1) d\theta$$

$$= \frac{1}{2} [\theta]_0^{\frac{\pi}{2}} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$

9). Show by double integration, the area between the parabolas y^2 = 4ax and x^2 = 4ay is $\frac{16}{3} a^2$

Sol: The P OI of given curves is A(0,0) and B(4a,4a). by taking a vertical strip parallel to y-axis. We get the area between the two parabolas as:



$$A = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} dx$$

$$= \int_{x=0}^{4a} [y]_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy$$

$$= \int_{x=0}^{4a} \left(2\sqrt{ax} - \frac{x^2}{4a}\right) dx$$

$$= \left[2\sqrt{a}.\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{12}\right]_0^{4a} = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2$$

$$\therefore \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{y=2\sqrt{ax}} dx = \frac{16}{3}a^2$$

Triple integrals:

Let f(x,y,z) be a function which is defined at all points in a finite region v in space. Let δx, δy, δzbe an elementary volume v enclosing of the point (x,y,z) thus the triple summation.

$$\lim_{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}} \sum\sum\sum f(x,y,z)\,\delta x, \delta y, \delta z$$

If it exists is written as $\iiint f(x,y,z)$ dx dydz which is called the triple integral of f(x,y,z) over the region v.

If the region v is bounded by the surfaces $x=x_1$, $x=x_2$, $y=y_1$, $y=y_2$, $z=z_1$, $z=z_2$ then

$$\iiint f(x,y,z)\text{dxdydz} = \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x,y,z)dx\,dy\,dz$$

Note:

(i) If x_1, x_2; y_1, y_2; z_1, z_2 are all constants then the order of integration is immaterial provide the limits of integration are changed accordingly.

i.e.

$$= \int_{z_1}^{z_2}\int_{y_1}^{y_2}\int_{x_1}^{x_2} f(x,y,z)dx\ dy\ dz$$

$$= \int_{y_1}^{y_2}\int_{z_1}^{z_2}\int_{x_1}^{x_2} f(x,y,z)dx\ dz\ dy$$

$$= \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)dz\ dy\ dx$$

(ii) If, however z_1, z_2 are functions of x and y and y_1, y_2 are functions of x while x_1 and x_2 are constants then the integration must be performed first w.r.to 'z' then w.r.to 'y' and finally w.r.to 'x'.

i.e.

$\iiint f(x,y,z)$dxdydz =

$$= \int_{x=a}^{x=b}\int_{y=\emptyset_1(x)}^{y=\emptyset_2(x)}\int_{z=\delta_1(x,y)}^{z=(x,y)} f(x,y,z)dz\ dy\ dx$$

10). Evaluate the following integrals:

(i) $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\ \ dz\ dy\ dx$

Sol $$\int_{x=0}^{1}\left\{\int_{y=0}^{\sqrt{1-x^2}}\left[\int_{z=0}^{\sqrt{1-x^2-y^2}} xyz\ \ dz\right] dy\right\} dx$$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{\sqrt{1-x^2}}\left[xy\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}}\right\} dy\ dx$$

$$= \int_{x=0}^{1}\left\{\int_{y=0}^{\sqrt{1-x^2}}\left[\frac{xy\,(1-x^2-y^2)}{2}\right]\right\} dy\ dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\left\{\int_{y=0}^{\sqrt{1-x^2}}[\,xy\ - x^3y - xy^3]\right\} dy\ dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\left[\frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{2}\int_{x=0}^{1} \frac{x(1-x^2)}{2} - \frac{x^3(1-x^2)}{2} - \frac{x(1-x^2)^2}{4} dx$$

$$= \frac{1}{2}\int_{0}^{1} \frac{x-x^3-x^3+x^5)}{2} - \frac{x(1-2x^2+x^4)}{4} dx$$

$$= \frac{1}{2}\int_{0}^{1} \frac{x-2x^3+x^5)}{2} - \frac{2x^3-x-x^5)}{4} dx$$

$$= \frac{1}{8}\int_0^1 x - 2x^3 + x^5 dx$$

$$= \frac{1}{8}\left[\frac{x^2}{2} - \frac{2x^4}{4} - \frac{x^6}{6}\right]_0^1$$

$$= \frac{1}{8}\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right] = \frac{1}{48}$$

$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\ dz\ dy\ dx = \frac{1}{48}$$

(ii) $\int_1^e \int_1^{logy} \int_1^{e^x} \log y\, dz\, dx\, dy$

Sol. I = $\int_{y=1}^{e} \int_{x=1}^{logy} \int_{z=1}^{e^x} \log z\, dz\, dx\, dy$

Consider $\int_{z=1}^{e^x} \log z\, dz = [\text{zlogz} - \text{z}]_1^{e^x}$

$= e^x \log e^x{}_e - e^x + 1$

$= xe^x - e^x + 1$

$= e^x(x-1) + 1$

I = $\int_{y=1}^{e} \int_{x=1}^{logy} \{(x-1)\, e^x + 1\}$ dx

Consider $\int_{x=1}^{x=logy} \{(x)\, e^x - e^x + 1\}$ dx

$= [xe^x - e^x - e^x + 1]_{x=1}^{logy}$

$$= [xe^x - 2e^x + 1]_1^{logy}$$

$$= [ylogy - 2y + logy] - [e - 2e + 1]$$

$$= (y+1)logy - 2y + (e-1)$$

$\therefore$ I = $\int_{y=1}^{e} ylogy + logy - 2y + (e-1)dy$

$$= \left[\frac{y^2}{2}logy - \frac{y^2}{4} + ylogy - y - y^2 + (e-1)y\right]_1^e$$

$$= [\frac{e^2}{2}loge - \frac{e^2}{4} + eloge - e - e^2 + (e-1)e] -$$

$$- [\frac{1}{2}log1 - \frac{1}{4} + log1 - 1 - 1 + e - 1]$$

$$= (\frac{e^2}{2} - \frac{e^2}{4} + e - 2e) - (-\frac{1}{4} - 3 - e)$$

$$= \frac{2e^2 - e^2 - 8e + 1 + 12}{4} = \frac{1}{4}[e^2 - 8e + 13]$$

$$\int_1^e \int_1^{logy} \int_1^{e^x} \log y \, dz \, dx \, dy = \frac{1}{4}[e^2 - 8e + 13]$$

MULTIPLE INTEGRALS

1. $\int_0^2 \int_0^x y\, dy\, dx$

   (a) $\frac{4}{3}$ (b) $\frac{4}{5}$ (c) $\frac{2}{3}$ (d) $\frac{2}{5}$

2. $\int_{x=0}^{a} \int_{y=0}^{b} (x^2 + y^2)$ dx dy

   (a) $\frac{1}{3}(a^2+b^2)$ (b) $\frac{a}{3}(a^2+b^2)$ (c) $\frac{b}{3}(a^2+b^2)$ (d) $\frac{ab}{3}(a^2+b^2)$

3. $\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{(1-x^2)(1-y^2)}}$

   (a) $\frac{\pi}{2}$ (b) $\frac{\pi^2}{2}$ (c) $\frac{\pi^2}{4}$ (d) $\frac{\pi}{4}$

9. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}$dxdy

   (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{8}$

10. $\int_0^\pi \int_0^{a\sin\theta} r\, dr\, d\theta$

    (a) $\frac{\pi a^2}{4}$ (b) $\frac{\pi a}{4}$ (c) $\frac{\pi a^2}{2}$ (d) $\frac{\pi a}{2}$

11. The iterated integral for $\int_{-1}^{1} \int_0^{1-x^2} f(x, y)\, dx\, dy$ after changing order of Integration is-------

    Ans: $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y) dx\, dy$

12. $\int_0^a \int_0^{\sqrt{x^2+y^2}} dx\, dy$ after changing to polar co-ordinates is

    (a) $\int_0^{\frac{\pi}{2}} \int_0^a r^3 \sin\theta\, dr\, d\theta$ (b) $\int_0^\pi \int_0^a r^2 \sin\theta\, dr\, d\theta$

    (c) $\int_0^{\frac{\pi}{2}} \int_0^a r \sin\theta\, dr\, d\theta$ (d) None

13. $\int_0^a \int_0^{\frac{a\sqrt{b^2-y^2}}{b}} xy\, dx\, dy$ after changing the order of integration is

    (a) $\int_0^a \int_0^{\frac{b\sqrt{a^2-x^2}}{b}} xy\, dx\, dy$ (b) $\int_0^{\frac{a\sqrt{b^2-y^2}}{b}} \int_0^b yx\, dy\, dx$

(c) $\int_0^a \int_0^b xy\,dx\,dy$ (d) None

14. $\int_0^1 \int_1^2 \int_2^3 x\,y\,z\,dx\,dy\,dz$

(a) $\frac{15}{2}$ (b) $\frac{15}{4}$ (c) $\frac{15}{9}$ (d) $\frac{15}{8}$

15. The area enclosed by the parabolas x^2 = y and y^2=x is

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\frac{3}{2}$ (d) $\frac{3}{4}$

16). The area of the region bounded by y^2 = 4ax and x^2 = 4ay is

(a) $\frac{4a^2}{3}$ (b) $\frac{8a^2}{3}$ (c) $\frac{16a^2}{3}$ (d) $\frac{25a^2}{3}$

17. the area of a plate in the form of a quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

(a) $\frac{\pi cb}{4}$ (b) $\frac{\pi cb}{2}$ (c) $\frac{\pi ab}{3}$ (d) None

18. $\int_0^1 \int_0^1 \int_0^1 x\,y\,x\,dx\,dy\,dz$

(a) $\frac{1}{3}$ (b) $\frac{1}{5}$ (c) $\frac{1}{8}$ (d) $\frac{1}{12}$

19. $\int_{-1}^1 \int_{-2}^2 \int_{-3}^3 dx\,dy\,dz$

(a) 12 (b) 24 (c) 48 (d) 36

20. The volume of tetrahedron formed by the surfaces x=0, y=0,z=0 and

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$ = 1 is

(a) $\frac{abc}{2}$ (b) $\frac{abc}{4}$ (c) $\frac{abc}{6}$ (d) $\frac{abc}{3}$

MULTIPLE INTEGRALS

1) (i) Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\, dx}{1+x^2+y^2}$

(ii) Evaluate $\int_0^5 \int_0^{x^2} x(x^2 + y^2)$ dx dy

2) (i) Evaluate $\iint (x^2 + y^2)$ dx dy in the positive quadratic for which

$x + y \leq 1$

(ii) Evaluate $\iint (x^2 + y^2)$ dx dy over the area bounded by the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

3) Evaluate $\int_0^{\frac{\pi}{4}} \int_0^{a\sin\theta} \frac{r\, dr\, d\theta}{\sqrt{a^2 - b^2}}$

4) Evaluate $\int_0^{4a} \int_{\frac{y^2}{4a}}^{y} \frac{x^2 - y^2}{x^2 + y^2}$ dx dy by changing into polar co- ordinates.

5) By changing the order of integration, evaluate $\int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy$

6) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx$

UNIT-IV

Vector Differentiation and Vector Operators

INTRODUCTION

In this chapter, vector differential calculus is considered, which extends the basic concepts of differential calculus, such as, continuity and differentiability to vector functions in a simple and natural way. Also, the new concepts of gradient, divergence and curl are introduced.

DIFFERENTIATION OF A VECTOR FUNCITON

Let S be a set of real numbers. Corresponding to each scalar t ε S, let there be associated a unique vector $\bar{f}$. Then $\bar{f}$ is said to be a vector (vector valued) function. S is called the domain of $\bar{f}$. We write $\bar{f}$ = $\bar{f}$(t).

Let $\bar{i}, \bar{j}, \bar{k}$ be three mutually perpendicular unit vectors in three dimensional space. We can write $\bar{f}$ = $\bar{f}$(t)= $f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$, where $f_1(t)$, $f_2(t)$, $f_3(t)$ are real valued functions (which are called components of $\bar{f}$). (we shall assume that $\bar{i},\bar{j},\bar{k}$ are constant vectors).

1. Derivative:

Let $\bar{f}$ be a vector function on an interval *I* and a ϵ *I*. then $Lt_{t\to a}\frac{\bar{f}(t)-\bar{f}(a)}{t-a}$, if exists, is called the derivative of $\bar{f}$ at a and is denoted by $\bar{f}^1$(a) or $\left(\frac{d\bar{f}}{dt}\right)$ at t = a. we also say that $\bar{f}$ is differentiable at t =a if $\bar{f}^1$(a) exists.

2. Higher order derivatives

Let $\bar{f}$ be differentiable on an interval *I* and $\bar{f}^1= \frac{d\bar{f}}{dt}$ be the derivative of $\bar{f}$. $Lt_{t\to a}\frac{\bar{f}^1(t)-\bar{f}^1(a)}{t-a}$ exists for every a Є I_1 . it is denoted by $\bar{f}^{11}= \frac{d^2\bar{f}}{dt^2}$.

Similarly we can define $\bar{f}^{111}(t)$ etc.

We now state some properties of differentiable functions (without proof)

(1) Derivative of a constant vector is $\bar{a}$.

If $\bar{a}$ and $\bar{b}$ are differentiable vector functions, then

(2). $\frac{d}{dt}(\bar{a}\pm\bar{b})=\frac{d\bar{a}}{dt}\pm\frac{d\bar{b}}{dt}$

(3). $\frac{d}{dt}(\bar{a}.\bar{b})=\frac{d\bar{a}}{dt}.\bar{b}+\bar{a}.\frac{d\bar{b}}{dt}$

(4). $\frac{d}{dt}(\bar{a} x \bar{b}) = \frac{d\bar{a}}{dt} x \bar{b} + \bar{a} x \frac{d\bar{b}}{dt}$

(5). If $\bar{f}$ is a differentiable vector function and ϕ is a scalar differential function, then $\frac{d}{dt}(\phi \bar{f}) = \phi \frac{d\bar{f}}{dt} + \frac{d\phi}{dt} \bar{f}$

(6). $\bar{f} = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$, where $f_1(t)$, $f_2(t)$, $f_3(t)$ are Cartesian components of the vector $\bar{f}$, then $\frac{d\bar{f}}{dt} = \frac{df_1}{dt}\bar{i} + \frac{df_2}{dt}\bar{j} + \frac{df_3}{dt}\bar{k}$

(7). The necessary and sufficient condition for $\bar{f}$(t) to be constant vector function is $\frac{d\bar{f}}{dt} = \bar{0}$.

3. Partial Derivatives

Partial differentiation for vector valued functions can be introduced as was done in the case of functions of real variables. Let $\bar{f}$ be a vector function of scalar variables *p, q, t*. Then we write $\bar{f} = \bar{f}(p,q,t)$. Treating *t* as a variable and *p,q* as constants, we define

$$Lt_{\delta t \to 0} \frac{\bar{f}(p,q,t+\delta t) - \bar{f}(p,q,t)}{\delta t}$$

If exists, as partial derivative of $\bar{f}$ w.r.t. *t* and is denot by $\frac{\partial \bar{f}}{\partial t}$

Similarly, we can define $\frac{\partial \bar{f}}{\partial p}, \frac{\partial \bar{f}}{\partial q}$ also. The following are some useful results on partial differentiation.

4. Properties

1) $\frac{\partial}{\partial t}(\phi \bar{a}) = \frac{\partial \phi}{\partial t}\bar{a} + \phi \frac{\partial \bar{a}}{\partial t}$

2). If λ is a constant, then $\frac{\partial}{\partial t}(\lambda \bar{a}) = \lambda \frac{\partial \bar{a}}{\partial t}$

3). If $\bar{c}$ is a constant vector, then $\frac{\partial}{\partial t}(\phi \bar{c}) = \bar{c}\frac{\partial \phi}{\partial t}$

4). $\frac{\partial}{\partial t}(\bar{a} \pm \bar{b}) = \frac{\partial \bar{a}}{\partial t} \pm \frac{\partial \bar{b}}{\partial t}$

5). $\frac{\partial}{\partial t}(\bar{a}.\bar{b}) = \frac{\partial \bar{a}}{\partial t}.\bar{b} + \bar{a}.\frac{\partial \bar{b}}{\partial t}$

6). $\frac{\partial}{\partial t}(\bar{a} x \bar{b}) = \frac{\partial \bar{a}}{\partial t} x \bar{b} + \bar{a} x \frac{\partial \bar{b}}{\partial t}$

7). Let $\bar{f} = f_1 \bar{i} + f_2 \bar{j} + f_3 \bar{k}$, where f_1, f_2, f_3 are differential scalar functions of more then one variable, Then $\frac{\partial \bar{f}}{\partial t} = \bar{i}\frac{\partial f_1}{\partial t} + \bar{j}\frac{\partial f_2}{\partial t} + \bar{k}\frac{\partial f_3}{\partial t}$ (treating $\bar{i}, \bar{j}, \bar{k}$ as fixed directions)

5. Higher order partial derivatives

Let $\bar{f} = \bar{f}(p,q,t)$. Then $\frac{\partial^2 \bar{f}}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \bar{f}}{\partial t}\right), \frac{\partial^2 \bar{f}}{\partial p \partial t} = \frac{\partial}{\partial p}\left(\frac{\partial \bar{f}}{\partial t}\right) etc.$

6.Scalar and vector point functions: Consider a region in three dimensional space. To each point $p(x,y,z)$, suppose we associate a unique real number (called scalar) say ϕ. This $\phi(x,y,z)$ is called a scalar point function. Scalar point function defined on the region. Similarly if to each point $p(x,y,z)$we associate a unique vector $\bar{f}(x,y,z)$ we associate a unique vector $\bar{f}(x,y,z)$. $\bar{f}$ is called a ***vector point function*.**

Examples:

For example take a heated solid. At each point p(x,y,z)of the solid, there will be temperature T(x,y,z). This T is a scalar point function.

Suppose a particle (or a very small insect) is tracing a path in space. When it occupies a position p(x,y,z) in space, it will be having some speed, say, v. This **speed**v is a scalar point function.

Consider a particle moving in space. At each point P on its path, the particle will be having a velocity $\bar{v}$ which is vector point function. Similarly, the acceleration of the particle is also a vector point function.

In a magnetic field, at any point P(x,y,z) there will be a magnetic force $\bar{f}(x,y,z)$. This is called magnetic force field. This is also an example of a vector point function. The students will come across several scalar and vector point functions in their respective subjects of study.

7. Tangent vector to a curve in space.

Consider an interval [a,b].

Let x = x(t),y=y(t),z=z(t)be continuous and derivable for a≤ t ≤b.

Then the set of all points (x(t),y(t),z(t)) is called a curve in a space.

Let A = (x(a),y(a),z(a)) and B = (x(b),y(b),z(b)). These A,B are called the end points of the curve. If A =B, the curve in said to be a closed curve.

Let P and Q be two neighbouring points on the curve.

Let $\overline{OP} = \bar{r}(t), \overline{OQ} = \bar{r}(t+\delta t) = \bar{r} + \delta\bar{r}. Then\ \delta\bar{r} = \overline{OQ} - \overline{OP} = \overline{PQ}$

Then $\frac{\delta r}{\delta t}$ is along the vector $\overline{PQ}$. As Q→P, PQ and hence $\frac{PQ}{\delta t}$ tends to be along the tangent to the curve at P.

Hence $\underset{\delta t \to 0}{lt} \frac{\delta \bar{r}}{\delta t} = \frac{d\bar{r}}{dt}$ will be a tangent vector to the curve at P. (This $\frac{d\bar{r}}{dt}$ may not be a unit vector)

Suppose arc length AP = s. if we take the parameter as the arc length parameter, we can observe that $\frac{d\bar{r}}{ds}$ is unit tangent vector at P to the curve.

VECTOR DIFFERENTIAL OPERATOR

Def. The vector differential operator ∇(read as del) is defined as

$\nabla \equiv \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$. This operator possesses properties analogous to those of ordinary vectors as well as differentiation operator. We will define now some quantities known as "**gradient**", "**divergence**" and "**curl**" involving this operator ∇. We must note that this operator has no meaning by itself unless it operates on some function suitably.

GRADIENT OF A SCALAR POINT FUNCTION

Let $\phi(x,y,z)$ be a scalar point function of position defined in some region of space. Then the vector function $\bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}$ is known as the gradient of ϕ or $\nabla\phi$

$$\nabla\phi = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)\phi = \bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}$$

Properties:

(1) If f and g are two scalar functions then grad(f ±g)= grad f ± grad g

(2) The necessary and sufficient condition for a scalar point function to be constant is that $\nabla f = \bar{o}$

(3) grad(fg) = f(grad g)+g(grad f)

(4) If c is a constant, grad (cf) = c(grad f)

(5) grad $\left(\frac{f}{g}\right)=\frac{g(grad\, f)-f(grad\, g)}{g^2}, (g\neq 0)$

(6) Let $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$. Then $d\bar{r}= (dx)\bar{i}+(dy)\bar{j}+(dz)\bar{k}$. if ϕ is any scalar point function, then $d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz$

$$=\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)$$

DIRECTIONAL DERIVATIVE

Let $\phi(x,y,z)$ be a scalar function defined throughout some region of space. Let this function have a value ϕ at a point P whose position vector referred to the origin O is OP = r. let $\phi+\Delta\phi$ be the value of the function at neighbouring point Q. If $\overline{OQ} = \bar{r} + \Delta r$. . Let Δr be the length of $\Delta\bar{r}$.

$\frac{\Delta\phi}{\Delta r}$ gives a measure of the rate at which ϕ change when we move from P to Q. then limiting value $\frac{\Delta\phi}{\Delta r}$ $as\ \Delta r \to 0$ is called the derivative of ϕ in the direction of PQ or simply directional derivative of ϕ at P and is denoted by $d\phi/dr$.

Theorem 1: The directional derivative of a scalar point function ϕ at a point P(x,y,z) in the direction of a unit vector e is equal to e. grad ϕ=e. $\nabla\phi$.

Level Surface

If a surface $\phi(x,y,z)= c$ be drawn through any point *P*(r), such that at each point on it, function has the same value as at P, then such a surface is called a level surface of the function ϕ through *P*.

e.g : equipotential or isothermal surface.

Theorem 2: $\nabla\phi$ at any point is a vector normal to the level surface $\phi(x,y,z)=c$ through that point, where *c* is a constant.

The physical interpretation of $\nabla\phi$

The gradient of a scalar function $\phi(x,y,z)$ at a point *P*(x,y,z) is a vector along the normal to the level surface $\phi(x,y,z) = c$ at *P* and is in increasing direction. Its magnitude is equal to the greatest rate of increase of ϕ. Greatest value of directional derivative of ϕ **at a point P = |grad ϕ|** at that point.

SOLVED EXAMPLES

Example 1: If $a=x+y+z$, $b= x^2+y^2+z^2$, $c = xy+yz+zx$, prove that [grad a, grad b, grad c] = 0.

Sol:- Given $a=x+y+z$ $\quad \frac{\partial a}{\partial x}=1, \frac{\partial a}{\partial y}=1, \frac{\partial a}{\partial z}=1$

Grad a = $\nabla a = \sum \bar{i}\frac{\partial a}{\partial x} = \bar{i}+\bar{j}+\bar{z}$

Given b= $x^2+y^2+z^2$ $\frac{\partial b}{\partial x}=2x, \frac{\partial b}{\partial y}=2y, \frac{\partial b}{\partial z}=2z$

Grad b = ∇b =$\bar{i}\frac{\partial b}{\partial x}+\bar{j}\frac{\partial b}{\partial y}+\bar{z}\frac{\partial b}{\partial z}=2x\bar{i}+2y\bar{j}+2z\bar{k}$

Again c = xy+yz+zx $\frac{\partial c}{\partial x}=y+z, \frac{\partial c}{\partial y}=z+x, \frac{\partial c}{\partial z}=y+x$

Grad c = $\bar{i}\frac{\partial c}{\partial x}+\bar{j}\frac{\partial c}{\partial y}+\bar{z}\frac{\partial c}{\partial z}=(y+z)\bar{i}+(z+x)\bar{j}+(x+y)\bar{k}$

[grad a, grad b, grad c] = $\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix}=0, (on\, simplification)$

[grad a, grad b, grad c] =0

Example 2: Show that $\nabla[f(r)] = \frac{f^{i}(r)}{r}\bar{r}$ where $\bar{r}= x\bar{i}+y\bar{j}+z\bar{k}$.

Sol:-since $\bar{r}= x\bar{i}+y\bar{j}+z\bar{k}$, we have $r^2= x^2+y^2+z^2$

Differentiating w.r.t. 'x' partially, we get

$2r\frac{\partial r}{\partial x}=2x \Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r}.Similarly \frac{\partial r}{\partial y}=\frac{y}{r}, \frac{\partial r}{\partial z}=\frac{z}{r}$

$\nabla[f(r)] =\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)f(r)=\sum \bar{i}f^{1}(r)\frac{\partial r}{\partial x}=\sum \bar{i}f^{1}(r)\frac{x}{r}$

$= \frac{f^{1}(r)}{r}\sum \bar{i}x=\frac{f^{1}(r)}{r}.\bar{r}$

Note : From the above result, $\nabla(\log r) = \frac{1}{r^2}r$

Example 3: Prove that $\nabla(r^n)= nr^{n-2}\bar{r}$.

Sol:- Let $\bar{r}= x\bar{i}+y\bar{j}+z\bar{k}$ and r = $|\bar{r}|$. Then we have $r^2 = x^2+y^2+z^2$ Differentiating w.r.t. x partially, we have

$2r\frac{\partial r}{\partial x}=2x \Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r}.Similarly=\frac{\partial r}{\partial y}=\frac{y}{r}\, and\, \frac{\partial r}{\partial z}=\frac{z}{r}$

$\nabla(r^n)= \sum \bar{i}\frac{\partial}{\partial x}(r^n)=\sum \bar{i}nr^{n-1}\frac{\partial r}{\partial x}=\sum \bar{i}nr^{n-1}\frac{x}{r}=nr^{n-2}\sum \bar{i}x=nr^{n-2}(\bar{r})$

Note : From the above result, we can have

(1). $\nabla\left(\frac{1}{r}\right)=-\frac{\bar{r}}{r^3}$, taking n = -1 (2) grad r = $\frac{\bar{r}}{r}$, taking n = 1

Example 4: Find the directional derivative of *f* = *xy*+*yz*+*zx* in the direction of vector $\bar{i}+2\bar{j}+2\bar{k}$ at the point (1,2,0).

Sol:- Given *f* = *xy*+*yz*+*zx*.

Grad f = $\bar{i}\frac{\partial f}{\partial x}+\bar{j}\frac{\partial f}{\partial y}+\bar{z}\frac{\partial f}{\partial z}=(y+z)\bar{i}+(z+x)\bar{j}+(x+y)\bar{k}$

If $\bar{e}$ is the unit vector in the direction of the vector $\bar{i}+2\bar{j}+2\bar{k}$, then

$$\bar{e}=\frac{\bar{i}+2\bar{j}+2\bar{k}}{\sqrt{1^2+2^2+2^2}}=\frac{1}{3}(\bar{i}+2\bar{j}+2\bar{k})$$

Directional derivative of *f* along the given direction $=\bar{e}.\nabla f$

$=\frac{1}{3}(\bar{i}+2\bar{j}+2\bar{k})[(y+2)\bar{i}+(z+x)\bar{j}+(x+y)\bar{k}]at\ (1,2,0)$

$=\frac{1}{3}[(y+z)+2(z+x)+2(x+y)](1,2,0)=\frac{10}{3}$

Example 5: Find the directional derivative of the function $xy^2+yz^2+zx^2$ along the tangent to the curve x =t, y = t^2, z = t^3 at the point (1,1,1).

Sol: - here *f* = $xy^2+yz^2+zx^2$

$\nabla f=\bar{i}\frac{\partial f}{\partial x}+\bar{j}\frac{\partial f}{\partial y}+\bar{k}\frac{\partial f}{\partial z}$ = (y2+2xy)j+(x2+2yz)k

At (1,1,1) , $\nabla f=3\bar{i}+3\bar{j}+3\bar{k}$

Let r be the position vector of any point on the curve x =t , y = t^2, z = t^3. then r = $x\bar{i} + y\bar{j} + z\bar{k} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$

$$\frac{\partial \bar{r}}{\partial t} = \bar{i} + 2t\bar{j} + 3t^2\bar{k} - (\bar{i} + 2\bar{j} + 3\bar{k}) \text{ at } (1,1,1)$$

We know that $\frac{\partial \bar{r}}{\partial t}$ is the vector along the tangent to the curve.

Unit vector along th: tangent $= \bar{e}\ \bar{t} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{1+2^2+3^2}} = \frac{\bar{i}+2\bar{j}+3\bar{k}}{\sqrt{14}}$

Directional derivative along the tangent = $\nabla f.\bar{e}$

$$= \frac{1}{\sqrt{14}}(\bar{i} + 2\bar{j} + 3\bar{k}).3(\bar{i} + \bar{j} + \bar{k})\ \frac{3}{\sqrt{14}}(1+2+3) = \frac{18}{\sqrt{14}}$$

Example 6: Find the directional derivative of the function $f = x^2-y^2+2z^2$ at the point P =(1,2,3) in the direction of the line $\overline{PQ}$ where Q = (5,0,4).

Sol:- The position vectors of P and Q with respect to the origin are OP = $\bar{i} + 2\bar{j} + 3\bar{k}$ and OQ = $5\bar{i} + 4\bar{k}$

$$\overline{PQ} = \overline{OQ} - \overline{OP} = 4\bar{i} - 2\bar{j} + \bar{k}$$

Let $\bar{e}$ be the unit vector in the direction of PQ. Then $\bar{e} = \frac{4\bar{i} - 2\bar{j} + \bar{k}}{\sqrt{21}}$

grad $f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = 2x\bar{i} - 2y\bar{j} + 4z\bar{k}$

The directional derivative of $\bar{f}$ at P (1,2,3) in the direction of PQ = $\bar{e}.\nabla f$

$$= \frac{1}{\sqrt{21}}(4\bar{i} - 2\bar{j} + \bar{k}).(2x\bar{i} - 2y\bar{j} + 4z\bar{k})\ \frac{1}{\sqrt{21}}(8x + 4y + 4z)_{at(1,2,3)} = \frac{1}{\sqrt{21}}(28)$$

Example 7: Find the greatest value of the directional derivative of the function f = x^2yz^3 at (2,1,-1).

Sol: we have

grad $f = \bar{i}\frac{\partial f}{\partial x}+\bar{j}\frac{\partial f}{\partial y}+\bar{k}\frac{\partial f}{\partial z} = 2xyz^3\bar{i}+x^2z^3\bar{j}+3x^2yz^2\bar{k} = -4\bar{i}-4\bar{j}+12\bar{k}$ at (2,1,-1).

Greatest value of the directional derivative of f = $|\nabla f| = \sqrt{16+16+144} = 4\sqrt{11}.$

Example 8: Find the directional derivative of xyz^2+xz at (1, 1 ,1) in a directional of the normal to the surface $3xy^2+y= z$ at (0,1,1).

Sol:- Let f(x, y, z) ≡ $3xy^2+y- z = 0$

Let us find the unti normal e to this surface at (,1,1). Then

$$\frac{\partial f}{\partial x} = 3y^2,\ \frac{\partial f}{\partial y} = 6xy+1,\frac{\partial f}{\partial z} = -1.$$

∇f = 3y2i+(6xy+1)j-k

$(\nabla f)_{(0,1,1)}$ = 3i+j-k = n

$$\bar{e} = \frac{\bar{n}}{|\bar{n}|} = \frac{3i+j-k}{\sqrt{9+1+1}} = \frac{3i+j-k}{\sqrt{11}}$$

Let g(x,y,z) = xyz^2+xz then

$$\frac{\partial g}{\partial x} = yz^2+z,\ \frac{\partial g}{\partial y} = xz^2,\frac{\partial g}{\partial z} = 2xy+x$$

$\nabla g=(yz^2+z)i+xz^2j+(2xyz+x)k$

And $[\nabla g]_{(1,1,1)}$ = 2i+j+3k

Directional derivative of the given function in the direction of $\bar{e}$ at (1,1,1) = $\nabla g.\bar{e}$

$$=(2i+j+3k).\left(\frac{3i+j-k}{\sqrt{11}}\right)=\frac{6+1-3}{\sqrt{11}}=\frac{4}{\sqrt{11}}$$

Example 9: Find the directional derivative of 2xy+z² at (1,-1,3) in the direction of $\bar{i}+2\bar{j}+3\bar{k}$.

Sol: Let $f = 2xy+z^2$ $\frac{\partial f}{\partial x}=2y, \frac{\partial f}{\partial y}=2x, \frac{\partial f}{\partial z}=2z.$

grad f = $\sum \bar{i}\frac{\partial f}{\partial x}=2y\bar{i}+2x\bar{j}+2z\bar{k}$ and (grad f)at (1,-1,3)= $-2\bar{i}+2\bar{j}+6\bar{k}$

given vector is $\bar{a}=\bar{i}+2\bar{j}+3\bar{k} \Rightarrow |\bar{a}|=\sqrt{1+4+9}=\sqrt{14}$

directional derivative of f in the direction of $\bar{a}$

$$\frac{\bar{a}.\nabla f}{|\bar{a}|}=\frac{(\bar{i}+2\bar{j}+3\bar{k})(-2\bar{i}+2\bar{j}+6\bar{k}).}{\sqrt{14}}=\frac{-2+4+18}{\sqrt{14}}=\frac{20}{\sqrt{14}}$$

Example 10: Find the directional derivative of $\phi = x^2yz+4xz^2$ at (1,-2,-1) in the direction 2i-j-2k.

Sol:- Given $\phi = x^2yz+4xz^2$

$$\frac{\partial \phi}{\partial x}=2xyz+4z^2, \frac{\partial \phi}{\partial y}=x^2z, \frac{\partial \phi}{\partial z}=x^2y+8xz.$$

Hence $\nabla\phi = \sum \bar{i}\frac{\partial \phi}{\partial x}=\bar{i}(2xyz+4z^2)+\bar{j}x^2z+\bar{k}(x^2y+8xz)$

$\nabla\phi$ at (1,-2,-1) = i(4+4)+j(-1)+k(-2-8)= 8i-j-10k.

The unit vector in the direction 2i-j-2k is

$$\bar{a}=\frac{2i-j-2k.}{\sqrt{4+1+4}}=\frac{1}{3}(2i-j-2k)$$

Required directional derivative along the given direction = ∇φ. $\bar{a}$

= (8i-j-10k). 1/3 (2i-j-2k)

= 1/3(16+1+20) = 37/3.

Example 11: If the temperature at any point in space is given by t = xy+yz+zx, find the direction in which temperature changes most rapidly with distance from the point (1,1,1) and determine the maximum rate of change.

Sol:- The greatest rate of increase of t at any point is given in magnitude and direction by ∇t.

We have ∇t = $\left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)(xy+yz+zx)$

= $\bar{i}(y+z)+\bar{j}(z+x)+\bar{k}(x+y)=2\bar{i}+2\bar{j}+2\bar{k}$ at (1,1,1)

Magnitude of this vector is $\sqrt{2^2+2^2+2^2}=\sqrt{12}=2\sqrt{3}$

Hence at the point (1,1,1) the temperature changes most rapidly in the direction given by the vector $2\bar{i}+2\bar{j}+2\bar{k}$ and greatest rate of increase = $2\sqrt{3}$.

Example 12: Findthe directional derivative of $\phi(x,y,z) = x^2yz+4xz^2$ at the point (1,-2,-1) in the direction of the normal to the surface $f(x,y,z) = x \log z - y^2$ at (-1,2,1).

Sol:- Given $\phi(x,y,z) = x^2yz+4xz^2$ at (1,-2,-1) and $f(x,y,z) = x \log z - y^2$ at (-1,2,1)

Now $\nabla\phi = \frac{\partial \phi}{\partial x}\bar{i} \mp \frac{\partial \phi}{\partial y}\bar{j} + \frac{\partial \phi}{\partial z}\bar{k}$

$= (2xyz + 4z^2)\bar{i} + (x^2 z)\bar{j} + (x^2 y + 8xz)\bar{k}$

$(\nabla\phi)_{(1,-2,-1)} = [2(1)(-2)(-1) + 4(-1)^2]i + [(1)^2(-1)\bar{j}] + [(1^2)(-2) + 8(-1)]\bar{k} ----(1)$

$= 8\bar{i} - \bar{j} - 10\bar{k}$

Unit normal to the surface

$f(x,y,z) = x \log z - y^2$ is $\frac{\nabla f}{|\nabla f|}$

now $\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = \log z\,\bar{i} + (-2y)\bar{j} + \frac{x}{z}\bar{k}$

at (-1,2,1), $\nabla f = \log(1)\bar{i} - 2(2)\bar{j} + \frac{-1}{1}\bar{k} = -4\bar{j} - \bar{k}$

$\frac{\nabla f}{|\nabla f|} = \frac{-4\bar{j} - \bar{k}.}{\sqrt{16+1}} = \frac{-4\bar{j} - \bar{k}.}{\sqrt{17}}$

Directional derivative $= \nabla\phi.\frac{\nabla f}{|\nabla f|}$

$= (8\bar{i} - \bar{j} - 10\bar{k}).\frac{-4\bar{j} - \bar{k}.}{\sqrt{17}} = \frac{4+10}{\sqrt{17}} = \frac{14}{\sqrt{17}}.$

Example 13: Find a unit normal vector to the given surface $x^2y+2xz = 4$ at the point (2,-2,3).

Sol:- Let the given surface be $f = x^2y+2xz - 4$

On differentiating,

$$\frac{\partial f}{\partial x} = 2xy + 2z, \frac{\partial f}{\partial y} = x^2, \frac{\partial f}{\partial z} = 2x.$$

grad $f = \sum \bar{i}\frac{\partial f}{\partial x} = \bar{i}(2xy + 2z) + \bar{j}x^2 + 2x\bar{k})$

(grad f) at (2,-2,3) = $\bar{i}(-8+6) + 4\bar{j} + 4\bar{k}) = -2\bar{i} + 4\bar{j} + 4\bar{k}$

grad (f) is the normal vector to the given surface at the given point.

Hence the required unit normal vector $\frac{\nabla f}{|\nabla f|} =$

$$\frac{2(-\bar{i} + 2\bar{j} + 2\bar{k}).}{2\sqrt{1+2^2+2^2}} = \frac{-\bar{i} + 2\bar{j} + 2\bar{k}}{3}$$

Example 14: Evaluate the angle between the normals to the surface $xy= z^2$ at the points (4,1,2) and (3,3,-3).

Sol:- given surface is $f(x,y,z) = xy= z^2$

Let $\bar{n}_1$ and $\bar{n}_2$ be the normals to this surface at (4,1,2) and (3,3,-3) respectively.

Differentiating partially, we get

$$\frac{\partial f}{\partial x} = y, \frac{\partial f}{\partial y} = x, \frac{\partial f}{\partial z} = -2z.$$

grad f = $y\bar{i} + x\bar{j} - 2z\bar{k}$

$\bar{n}_1$= (grad f) at (4,1,2) $= \bar{i} + 4\bar{j} - 4\bar{k}$

$\bar{n}_2$= (grad f) at (3,3,-3) = $3\bar{i} + 3\bar{j} + 6\bar{k}$

Let θ be the angle between the two normals.

$$\cos\theta = \frac{\bar{n}_1 \bar{n}_2}{|\bar{n}_1 \bar{n}_2|} = \frac{(i+4j-4k)}{\sqrt{1+16+16}} \cdot \frac{(3i+3j+6k)}{\sqrt{9+9+36}}$$

$$\frac{(3+12-24)}{\sqrt{33}\sqrt{54}} = \frac{-9}{\sqrt{33}\sqrt{54}}$$

Example 15: Find a unit normal vector to the surface $x^2+y^2+2z^2 = 26$ at the point (2, 2 ,3).

Sol:- Let the given surface be $f(x,y,z) \equiv x^2+y^2+2z^2 - 26=0$. Then

$$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = 4z.$$

grad f = $\sum \bar{i} \frac{\partial f}{\partial x}$ = 2xi+2yj+4zk

normal vector at(2,2,3) = $[\nabla f]_{(2,2,3)}$ = 4I+4J+12K

unit normal vector = $\frac{\nabla f}{|\nabla f|} = \frac{4(\bar{i} + \bar{j} + 3\bar{k})}{4\sqrt{11}} = \frac{\bar{i} + \bar{j} + 3\bar{k}}{\sqrt{11}}$

Example 16: Find the values of a and b so that the surfaces ax^2-byz = (a+2)x and $4x^2y+z^3$= 4 may intersect orthogonally at the point (1, -1,2).

(or) Find the constants a and b so that surface ax^2-byz=(a+2)x will orthogonal to $4x^2y+z^3$=4 at the point (1,-1,2).

Sol:- let the given surfaces be f(x,y,z) = ax^2-byz - (a+2)x-------------(1)

And g(x,y,z) = $4x^2y+z^3$- 4------------(2)

Given the two surfaces meet at the point (1,-1,2).

Substituting the point in (1), we get

$$a+2b-(a+2) = 0 \Rightarrow b=1$$

now $\frac{\partial f}{\partial x} = 2ax - (a+2), \frac{\partial f}{\partial y} = -bz =, \frac{\partial f}{\partial z} = -by.$

$\nabla f = \sum \bar{i}\frac{\partial f}{\partial x} =$ [(2a-(a+2)]i-2bj+bk = (a-2)i-2bj+bk

= (a-2)i-2j+k = $\bar{n}_1$, normal vector to surface 1.

Also $\frac{\partial g}{\partial x} = 8xy, \frac{\partial g}{\partial y} = 4x^2, \frac{\partial g}{\partial z} = 3z^2.$

$\nabla g = \sum \bar{i}\frac{\partial g}{\partial x} =$ 8xyi+4x^2j+3z^2k

$(\nabla g)_{(1,-1,2)}$ = -8i+4j+12k = $\bar{n}_2$, normal vector to surface 2.

Given the surfaces f(x,y,z), g(x,y,z) are orthogonal at the point (1,-1,2).

$[\overline{\nabla} f][\overline{\nabla} g] = 0 \Rightarrow$ ((a-2)i-2j+k). (-8i+4j+12k)=0

$\Rightarrow$-81+16-8+12 $\Rightarrow$ a =5/2

Hence a = 5/2 and b=1.

Example 17: Find a unit normal vector to the surface z= x^2+y^2 at (-1,-2,5)

Sol:- let the given surface be f = x^2+y^2-z

$$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, \frac{\partial f}{\partial z} = -1.$$

grad f = $\nabla f = \sum \bar{i}\frac{\partial f}{\partial x} =$ 2xi+2yj-k

(∇f) at (-1,-2,5)= -2i-4j-k

∇f is the normal vector to the given surface.

Hence the required unit normal vector = $\frac{\nabla f}{|\nabla f|}$ =

$$\frac{-2i-4j-k}{\sqrt{(-2)^2+(-4)^2+(-1)^2}} = \frac{-2i-4j-k}{\sqrt{21}} = -\frac{1}{\sqrt{21}}(2i+4j+k)$$

Example 18: Find the angle of intersection of the spheres $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$ at the point (4,-3,2).

Sol:- Let f = $x^2+y^2+z^2-29$ and g = $x^2+y^2+z^2+4x-6y-8z-47$

Then grad f= $\bar{i}\frac{\partial f}{\partial x}+\bar{j}\frac{\partial f}{\partial y}+\bar{k}\frac{\partial f}{\partial z}=2x\bar{i}+2y\bar{j}+2z\bar{k}$ and

grad g = $(2x+4)\bar{i}+(2y-6)\bar{j}+(2z-8)\bar{k}$

The angle between two surfaces at a point is the angle between the normals to the surfaces at that point.

Let $\bar{n}_1$= (grad *f*) at (4,-3,2) =$8\bar{i}-6\bar{j}+4\bar{k}$

$\bar{n}_2$= (grad *f*) at (4,-3,2) = $12\bar{i}-12\bar{j}-4\bar{k}$

The vectors $\bar{n}_1$ and $\bar{n}_2$ are along the normals to the two surfaces at (4,-3,2). Let θ be the angle between the surfaces. Then

$$\text{Cos }\theta=\frac{\bar{n}_1\,\bar{n}_2}{|\bar{n}_1\,\bar{n}_2|}=.\frac{152}{\sqrt{116}\sqrt{304}} \quad \therefore \theta=\cos^{-1}\left(\sqrt{\frac{19}{29}}\right)$$

Example 19: Find the angle between the surfaces $x^2+y^2+z^2=9$, and $z=x^2+y^2-3$ at point (2,-1,2).

Sol:- Let $\phi_1=x^2+y^2+z^2-9=0$ and $\phi_2=x^2+y^2-z-3=0$ be the given surfaces. Then

$\nabla\phi_1 = 2xi+2yj+2zk$ and $\nabla\phi_2 = 2xi+2yj-k$

Let $\bar{n}_1 = \nabla\phi_1$ at(2,-1,2)= 4i-2j+4k and

$\bar{n}_2 = \nabla\phi_2$ at (2,-1,2) = 4i-2j-k

The vectors $\bar{n}_1$ and $\bar{n}_2$ are along the normals to the two surfaces at the point (2,-1,2). Let θ be the angle between the surfaces. Then

$$\text{Cos}\ \theta = \frac{\bar{n}_1\,\bar{n}_2}{|\bar{n}_1\,\bar{n}_2|} = \frac{(4i-2j+4k)}{\sqrt{16+4+16}}.\frac{(4i-2j-k)}{\sqrt{16+4+16}} = \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right).$$

Example 20: If $\bar{a}$ is constant vector then prove that grad $(\bar{a}\,.\,\bar{r}) = \bar{a}$

Sol: Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$, where a_1, a_2, a_3 are constants.

$$\bar{a}\,.\,\bar{r} = \left(a_1\bar{i} + a_2\bar{j} + a_3\bar{k}\right).\left(x\bar{i} + y\bar{j} + z\bar{k}\right) = a_1x + a_2y + a_3z$$

$$\frac{\partial}{\partial x}(\bar{a}.\bar{r}) = a_1, \frac{\partial}{\partial y}(\bar{a}.\bar{r}) = a_2, \frac{\partial}{\partial z}(\bar{a}.\bar{r}) = a_3$$

grad $(\bar{a}\,.\,\bar{r}) = a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$

Example 21: If $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$, find φ.

Sol:- we know that $\nabla\phi = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

Given that $\nabla\phi = yz\bar{i} + zx\bar{j} + xy\bar{k}$

Comparing the corresponding coefficients, we have

$\frac{\partial \phi}{\partial x} = yz, \frac{\partial \phi}{\partial y} = zx, \frac{\partial \phi}{\partial z} = xy$

Integrating partially w.r.t. x,y,z, respectively, we get

ϕ= xyz + a constant independent of x.

ϕ= xyz + a constant independent of y.

ϕ= xyz + a constant independent of z.

Here a possible form of ϕ is ϕ= xyz+a constant.

DIVERGENCE OF A VECTOR

Let $\bar{f}$ be any continuously differentiable vector point function. Then $\bar{i}.\frac{\partial \bar{f}}{\partial x} + \bar{j}.\frac{\partial \bar{f}}{\partial y} + \bar{k}.\frac{\partial \bar{f}}{\partial z}$ is called the divergence of $\bar{f}$ and is written as div $\bar{f}$.

i.e div $\bar{f} = \bar{i}.\frac{\partial \bar{f}}{\partial x} + \bar{j}.\frac{\partial \bar{f}}{\partial y} + \bar{k}.\frac{\partial \bar{f}}{\partial z} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right).\bar{f}$

hence we can write div $\bar{f}$ as

div $\bar{f} = \nabla.\bar{f}$

This is a scalar point function.

Theorem 1: If the vector $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$, then div $\bar{f}$ = $\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Prof: Given $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$\frac{\partial \bar{f}}{\partial x} = \bar{i}\frac{\partial f_1}{\partial x} + \bar{j}\frac{\partial f_2}{\partial x} + \bar{k}\frac{\partial f_3}{\partial x}$

Also $\bar{i}.\frac{\partial \bar{f}}{\partial x} = \frac{\partial f_1}{\partial x}$. Similarly $\bar{j}.\frac{\partial \bar{f}}{\partial y} = \frac{\partial f_2}{\partial y}$ and $\bar{k}.\frac{\partial \bar{f}}{\partial z} = \frac{\partial f_3}{\partial z}$

We have div $\bar{f} = \sum \bar{i}.\left(\frac{\partial \bar{f}}{\partial x}\right) = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

Note : If $\bar{f}$ is a constant vector then $\frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial y}, \frac{\partial f_3}{\partial z}$ are zeros.

div $\bar{f}$=0 for a constant vector $\bar{f}$.

Theorem 2: div $(\bar{f} \pm \bar{g}) = div\,\bar{f} \pm div\,\bar{g}$

Proof: div $(\bar{f} \pm \bar{g}) = \sum \bar{i}.\frac{\partial}{\partial x}(\bar{f}) \pm \sum \bar{i}.\frac{\partial}{\partial x}(\bar{g}) = div\,\bar{f} \pm div\,\bar{g}$.

Note: If ϕ is a scalar function and $\bar{f}$ is a vector function, then

(i). $(\bar{a}.\nabla)\phi = \left[\bar{a}\left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\right]\phi$

$= \left[(\bar{a}.\bar{i})\frac{\partial}{\partial x} + (\bar{a}.\bar{j})\frac{\partial}{\partial y} + (\bar{a}.\bar{k})\frac{\partial}{\partial z}\right]\phi$

$= \left[(\bar{a}.\bar{i})\frac{\partial \phi}{\partial x} + (\bar{a}.\bar{j})\frac{\partial \phi}{\partial y} + (\bar{a}.\bar{k})\frac{\partial \phi}{\partial z}\right]$

$= \sum (\bar{a}.\bar{i})\frac{\partial \phi}{\partial x}$. and

(ii). $(\bar{a}.\nabla)\bar{f} = \sum (\bar{a}.\bar{i})\frac{\partial \bar{f}}{\partial x}$. by proceeding as in (i) [simply replace ϕ by $\bar{f}$ in (i)].

SOLENOIDAL VECTOR

A vector point function $\bar{f}$ is said to be $\bar{f}$ solenoidal if div $\bar{f}$=0.

Physical interpretation of divergence:

Depending upon $\bar{f}$ in a physical problem, we can interpret div $\bar{f}$ (= $\nabla . \bar{f}$).

Suppose $\bar{F}$(x,y,z,t) is the velocity of a fluid at a point(x,y,z) and time 't'. though time has no role in computing divergence, it is considered here because velocity vector depends on time.

Imagin a small rectangular box within the fluid as shown in the figure. We would like to measure the rate per unit volume at which the fluid flows out at any given time. The divergence of $\bar{F}$ measures the outward flow or expansions of the fluid from their point at any time. This gives a physical interpretation of the divergence.

Similar meanings are to be understood with respect to divergence of vectors $\bar{f}$ from other branches. A detailed elementary interpretation can be seen in standard books on fluid dynamics, electricity and magnetism etc.

SOLVED EXAMPLES

Example 1: If $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$ find div $\bar{f}$ at(1, -1, 1).

Sol:- $\bar{f} = xy^2\bar{i} + 2x^2yz\bar{j} - 3yz^2\bar{k}$. Then

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) + \frac{\partial}{\partial z}(-3yz^2) = y^2+2x^2z-6yz$

(div $\bar{f}$) at (1, -1, 1) = 1+2+6 =9

Example 2: find div $\bar{f}$ = grad($x^3+y^3+z^3-3xyz$)

Sol:- Let $\phi = x^3+y^3+z^3-3xyz$. Then

$$\frac{\partial \phi}{\partial x} = 3x^2 - 3yz, \frac{\partial \phi}{\partial y} = 3y^2 - 3zx, \frac{\partial \phi}{\partial z} = 3z^2 - 3xy$$

grad $\phi = \bar{i}\frac{\partial \phi}{\partial x} + \bar{j}\frac{\partial \phi}{\partial y} + \bar{k}\frac{\partial \phi}{\partial z} = 3[(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}]$

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = \frac{\partial}{\partial x}[3(x^2 - yz)].\frac{\partial}{\partial y}[3(y^2 - zx)].\frac{\partial}{\partial z}[3(z^2 - xy)]$

= 3(2*x*)+3(2*y*)+3(2*z*) = 6(*x*+*y*+*z*)

Example 3: If $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k}$ is solenoidal, find *P*.

Sol:- Let $\bar{f} = (x+3y)\bar{i} + (y-2z)\bar{j} + (x+pz)\bar{k} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

We have $\frac{\partial f_1}{\partial x} = 1, \frac{\partial f_2}{\partial y} = 1, \frac{\partial f_3}{\partial z} = p$

div $\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$ = 1+1+p =2+p since $\bar{f}$ is solenoidal, we have div $\bar{f} = 0 \rightarrow$ p = -2

Example 4: Find div $\bar{f} = r^n \bar{r}$. Find n if it is solenoidal?

Sol: Given $\bar{f} = r^n \bar{r}$. where $\bar{f} = \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and $r = |\bar{r}|$

We have $r^2 = x^2+y^2+z^2$

Differentiating partially w.r.t. x , we get

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$, and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$\bar{f} = r^n\,(x\bar{i} + y\bar{j} + z\bar{k})$

$$\text{div } \bar{f} = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= nr^{n-1}\frac{\partial r}{\partial x}x + nr^{n-1}\frac{\partial r}{\partial y}y + nr^{n-1}\frac{\partial r}{\partial z}z + r^n$$

$$= nr^{n-1}\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right] + 3r^n = nr^{n-1}\frac{(r^2)}{r} + 3r^n = nr^n + 3r^n = (n+3)r^n$$

Let $\bar{f} = r^n \bar{r}$. be solenoidal. Then div $\bar{f} = 0$

$(n+3)r^n = 0 \Rightarrow n = -3$

Example 5: Evaluate $\nabla.\left(\frac{\bar{r}}{r^3}\right)$ where $\bar{r} = xi + yj + zk$ and $r = |\bar{r}|$.

Sol:- We have

r = xi+yj+zk and $r = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \text{and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \frac{\bar{r}}{r^3} = \bar{r}.\ r^{-3} = r^{-3}xi + r^{-3}yj + r^{-3}zk = f_1 i + f_2 j + f_3 k$$

Hence $\nabla.\left(\frac{\bar{r}}{r^3}\right)=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$

We have f_1= $r^{-3}x$ ➔ $\frac{\partial f_1}{\partial x}=r^{-3}.1+x(-3)r^{-4}.\frac{\partial r}{\partial x}$

$\nabla\left(\frac{\bar{r}}{r^3}\right)=\sum\bar{i}.\frac{\partial f_1}{\partial x}=3r^{-3}-3r^{-5}\sum x^2$

$= 3r^{-3}-3r^{-5}r^2 = 3r^{-3}-3r^{-3} =0$

Example 6: Find div $\bar{r}.$ where $\bar{r}.= x\bar{i}+y\bar{j}+z\bar{k}$

Sol:- We have $\bar{r}.= x\bar{i}+y\bar{j}+z\bar{k}=f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$

div $\bar{r}.=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)=1+1+1=3$

CURL OF A VECTOR

Def: Let $\bar{f}$ be any continuously differentiable vector point function. Then the vector function defined by $\bar{i}x\frac{\partial\bar{f}}{\partial x}+\bar{j}x\frac{\partial\bar{f}}{\partial y}+\bar{k}x\frac{\partial\bar{f}}{\partial z}$ is called curl of $\bar{f}$ and is denoted by curl $\bar{f}$ or (∇x$\bar{f}$).

Curl $\bar{f}$ = $\bar{i}x\frac{\partial\bar{f}}{\partial x}+\bar{j}x\frac{\partial\bar{f}}{\partial y}+\bar{k}x\frac{\partial\bar{f}}{\partial z}=\sum\left(\bar{i}x\frac{\partial\bar{f}}{\partial x}\right)$

Theorem 1: If $\bar{f}$ is differentiable vector point function given by $\bar{f}=f_1\bar{i}+f_2\bar{j}+f_3\bar{k}$ then curl $\bar{f}$ =

$$\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)\bar{i}+\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)\bar{j}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\bar{k}$$

Proof : curl $\bar{f}$ = $\sum\bar{i}x\frac{\partial}{\partial x}(\bar{f})=\sum\bar{i}x\frac{\partial}{\partial x}(f_1\bar{i}+f_2\bar{j}+f_3\bar{k})=\sum\left(\frac{\partial f_2}{\partial x}\bar{k}-\frac{\partial f_3}{\partial x}\bar{j}\right)$

$$=\left(\frac{\partial f_2}{\partial x}\bar{k}-\frac{\partial f_3}{\partial x}\bar{j}\right)+\left(\frac{\partial f_3}{\partial y}\bar{i}-\frac{\partial f_1}{\partial y}\bar{k}\right)+\left(\frac{\partial f_1}{\partial z}\bar{j}-\frac{\partial f_2}{\partial z}\bar{i}\right)$$

$$=\bar{i}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)+\bar{j}\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)+\bar{k}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)$$

Note : (1) The above expression for curl $\bar{f}$ can be remembered easily through the representation.

$$\textbf{curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \nabla \mathbf{x}\, \bar{f}$$

note : (2) If $\bar{f}$ is a constant vector then curl $\bar{f} = \bar{o}$.

Theorem 2: curl $\left(\bar{a} \pm \bar{b}\right) = curl\, \bar{a} \pm curl\, \bar{b}$

Proof: curl$\left(\bar{a} \pm \bar{b}\right) = \sum \bar{i} x \frac{\partial}{\partial x}\left(\bar{a} \pm \bar{b}\right)$

$$=\sum ix\left(\frac{\partial \bar{a}}{\partial x} \pm \frac{\partial \bar{b}}{\partial x}\right)=\sum ix\frac{\partial \bar{a}}{\partial x} \pm \sum ix \frac{\partial \bar{b}}{\partial x}$$

$$= curl\, \bar{a} \pm curl\, \bar{b}$$

1. Physical Interpretation of curl

If $\bar{w}$ is the angular velocity of a rigid body rotating about a fixed axis and $\bar{v}$ is the velocity of any point P(x,y,z) on the body, then $\bar{w}$ = ½ curl $\bar{v}$. Thus the angular velocity of rotation at any point is equal to half the curl of velocity vector. This justifies the use of the word "curl of a vector".

2. Irrotational Motion, Irrotational Vector

Any motion in which curl of the velocity vector is a null vector i.e curl $\bar{v}=\bar{0}$ is said to be Irrotational.

Def: A vector $\bar{f}$ is said to be Irrotational if curl $\bar{f} = \bar{0}$.

If $\bar{f}$ is Irrotational, there will always exist a scalar function φ(x,y,z) such that $\bar{f}$=grad ϕ. This is called scalar potential of $\bar{f}$.

It is easy to prove that, if $\bar{f}$ = grad ϕ, then curl $\bar{f}$= 0.

Hence ∇x$\bar{f}$ = 0 ⇔ there exists a scalar function ϕ such that $\bar{f}$= ∇ϕ.

This idea is useful when we study the "work done by a force" later.

SOLVED EXAMPLES

Example 1: if $\bar{f}= xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$ find curl $\bar{f}$ at the point (1,-1,1).

Sol:- Let $\bar{f}= xy^2\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$. Then

$$\text{curl } \bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$=$

$$\bar{i}\left(\frac{\partial}{\partial y}(-3yz^2)-\frac{\partial}{\partial z}(2x^2yz)\right)+\bar{j}\left(\frac{\partial}{\partial z}(xy^2)-\frac{\partial}{\partial x}(-3yz^2)\right)+\bar{k}\left(\frac{\partial}{\partial x}(2x^2yz)-\frac{\partial}{\partial y}(xy^2)\right)$$

$=\bar{i}\left(-3z^2-2x^2z\right)+\bar{j}(0-0)+\bar{k}(4xyz-2xy)$

= curl $\bar{f}$= at (1,-1,1) = $-\bar{i}-2\bar{k}.$

Example 2: Find curl $\bar{f}$ where $\bar{f}$= grad($x^3+y^3+z^3-3xyz$)

Sol:- Let $\phi = x^3+y^3+z^3-3xyz$. Then

grad ϕ= $\sum \bar{i}\frac{\partial \phi}{\partial x}=3(x^2-yz)\bar{i}+3(y^2-zx)\bar{j}+3(z^2-xy)\bar{k}$

curl grad ϕ= ∇x grad ϕ= $3\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$

$=3[\bar{i}(-x+x)-\bar{j}(-y+y)+\bar{k}(-z+z)]=\bar{0}$

curl $\bar{f}=\bar{0}$.

Note: We can prove in general that curl (grad ϕ)=$\bar{0}$.(i.e) grad ϕ is always irrotational.

Example 3: Prove that if $\bar{r}$ is the position vector of an point in space, then $r^n\bar{r}$ is Irrotational. (or) Show that curl $(r^n\bar{r})=0$

Sol:- Let $\bar{r}= x\bar{i}+y\bar{j}+z\bar{k}$ and r = $|\bar{r}|$ $\therefore r^2=x^2+y^2+z^2$.

Differentiating partially w.r.t. 'x' partially, we get

$$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r},$$

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}, and\ \frac{\partial r}{\partial z} = \frac{z}{r}$

$r^n = |\bar{r}|\ \ r^n(x\bar{i} + y\bar{j} + z\bar{k})$

$$x(r^n\bar{r}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xr^n & yr^n & zr^n \end{vmatrix}$$

$$= \bar{i}\left(\frac{\partial}{\partial y}(r^n z) - \frac{\partial}{\partial z}(r^n y)\right) + \bar{j}\left(\frac{\partial}{\partial z}(r^n x) - \frac{\partial}{\partial x}(r^n z)\right) + \bar{k}\left(\frac{\partial}{\partial x}(r^n y) - \frac{\partial}{\partial y}(r^n x)\right)$$

$$= \sum \bar{i}\left\{znr^{n-1}\frac{\partial r}{\partial y} - ynr^{n-1}\frac{\partial r}{\partial z}\right\} = nr^{n-1}\sum \bar{i}\left\{z\left(\frac{y}{r}\right) - z\left(\frac{z}{r}\right)\right\}$$

$$nr^{n-2}[(zy - yz)\bar{i} + (xz - zx)\bar{j} + (xy - yz)\bar{k}]$$

$$nr^{n-2}[0\bar{i} + 0j + 0\bar{k}] = nr^{n-2}[\bar{0}] = \bar{0}$$

Hence $r^n\bar{r}$ is Irrotational.

Example 4: Prove that curl $\bar{r} = \bar{0}$

Sol:- Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

curl $\bar{r} = \sum \bar{i} x \frac{\partial}{\partial x}(\bar{r}) = \sum (\bar{i} x \bar{i}) = \bar{0} + \bar{0} = \bar{0}$

$\bar{r}$ is Irrotational vector.

Example 5: If $\bar{a}$ is a constant vector, prove that curl $\left(\frac{\bar{a} x \bar{r}}{r^3}\right) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{a}.\bar{r}).$

Sol:- We have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\frac{\partial \bar{r}}{\partial x} = \bar{i},\ \frac{\partial \bar{r}}{\partial y} = \bar{j},\ \frac{\partial \bar{r}}{\partial z} = \bar{k}$$

If $|\bar{r}| = r$ then $r^2 = x^2 + y^2 + z^2$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, and \frac{\partial r}{\partial z} = \frac{z}{r}$$

curl $\left(\frac{\bar{a} x \bar{r}}{r^3}\right) = \sum \bar{i} x \frac{\partial}{\partial x}\left(\frac{\bar{a} x \bar{r}}{r^3}\right)$

Now $\frac{\partial}{\partial x}\left(\frac{\bar{a} x \bar{r}}{r^3}\right) = \bar{a} x \frac{\partial}{\partial x}\left(\frac{\bar{r}}{r^3}\right) = \bar{a} x\left[\frac{1}{r^3}\frac{\partial \bar{r}}{\partial x} - \frac{3}{r^4}\frac{\partial r}{\partial x}\bar{r}\right]$

$$= \bar{a} x\left[\frac{1}{r^3}\bar{i} - \frac{3}{r^5} x\bar{r}\right] = \frac{\bar{a} x \bar{i}}{r^3} - \frac{3x(\bar{a} . x\bar{r})}{r^5}.$$

$$\therefore\ ix\frac{\partial}{\partial x}\left(\frac{\bar{a} x \bar{r}}{r^3}\right) = \bar{i} x\left[\frac{\bar{a} x \bar{i}}{r^3} - \frac{3x}{r^5}(\bar{a} x \bar{r})\right] = \frac{\bar{i} x(\bar{a} x \bar{i})}{r^3} - \frac{3x}{r^5}\bar{i} x(\bar{a} x \bar{r})$$

$$= \frac{(\bar{i}.\bar{i})\bar{a} - (\bar{i}.\bar{a})i}{r^3} - \frac{3x}{r^5}[(\bar{i}.\bar{r})\bar{a} - (i.a)\bar{r}]$$

Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$. Then $\bar{i} . \bar{a} = a_1$, etc.

$$\therefore\ ix\frac{\partial}{\partial x}\left(\frac{\bar{a} x \bar{r}}{r^3}\right) = \sum \frac{(\bar{a} - a_1 i)}{r^3} - \frac{3x}{r^3}(x\bar{a} - a_1\bar{r})$$

$$\therefore\ \sum ix\frac{\partial}{\partial x}\left(\frac{\bar{a} x \bar{r}}{r^3}\right) = \sum \frac{\bar{a} - a_1\bar{i}}{r^3} - \frac{3}{r^5}\sum (x^2\bar{a} - a_1 x\bar{r})$$

$$= \frac{3\bar{a} - \bar{a}}{r^3} - \frac{3\bar{a}}{r^5}(r^2) + \frac{3\bar{r}}{r^5}(a_1 x + a_2 y + a_3 z)$$

$$= \frac{2\bar{a}}{r^3} - \frac{3\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{r}.\bar{a}) = -\frac{\bar{a}}{r^3} + \frac{3\bar{r}}{r^5}(\bar{r}.\bar{a})$$

Example 6: Show that the vector $(x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ is irrotational and find its scalar potential.

Sol: let $\bar{f} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$

$$\text{Then curl } \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix} = \sum i(-x + x) = \bar{0}$$

$\bar{f}$ is Irrotational. Then there exists ϕ such that $\bar{f} = \nabla\phi$.

$$\Rightarrow \bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}= (x^2-yz)\bar{i}+(y^2-zx)\,\bar{j}+(z^2-xy)\,\bar{k}$$

Comparing components, we get

$$\frac{\partial\phi}{\partial x}=x^2-yz \Rightarrow \phi=\int\left(x^2-yz\right)dx=\frac{x^3}{3}-xyz+f_1(y,z)......(1)$$

$$\frac{\partial\phi}{\partial y}=y^2-zx \Rightarrow \phi=\frac{y^3}{3}-xyz+f_2(z,x)......(2)$$

$$\frac{\partial\phi}{\partial z}=z^2-xy \Rightarrow \phi=\frac{z^3}{3}-xyz+f_3(x,y)......(3)$$

From (1), (2),(3), $\phi=\frac{x^3+y^3+z^3}{3}-xyz$

$$\therefore \quad \phi=\frac{1}{3}(x^3+y^3+z^3)-xyz+cons\tan t$$

Which is the required scalar potential.

Example 7: Find constants a,b and c if the vector $\bar{f}=$ $(2x+3y+az)\bar{i}+(bx+2y+3z)\,\bar{j}+(2x+cy+3z)\,\bar{k}$ is Irrotational.

Sol:- Given $\bar{f}=(2x+3y+az)\bar{i}+(bx+2y+3z)\,\bar{j}+(2x+cy+3z)\,\bar{k}$

$$\text{Curl }\bar{f}=\begin{vmatrix}\bar{i} & \bar{j} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2x+3y+az & bx+2y+3z & 2x+cy+3z\end{vmatrix}=$$

$$(c-3)\bar{i}+(2-a)\,\bar{j}+(b-3)\,\bar{k}$$

If the vector is Irrotational then curl $\bar{f}=\bar{0}$

c-3 = 2-a=0, b-3 = 0 $\Rightarrow$c=3, a=2, b=3.

Example 8: If f(r) is differentiable, show that curl { $\bar{r}$ f(r)} = $\bar{0}$ where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

Sol: r = $\bar{r} = \sqrt{x^2 + y^2 + z^2}$ $r^2 = x^2+y^2+z^2$

$\Rightarrow 2r\,\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$, similarly $\frac{\partial r}{\partial y} = \frac{y}{r}, and\ \frac{\partial r}{\partial z} = \frac{z}{r}$

curl{$\bar{r}$ f(r)}= curl{f(r)($x\bar{i} + y\bar{j} + z\bar{k}$)}= curl $(x.f(r)\bar{i} + y.f(r)\,\bar{j} + z.f(r)\,\bar{k}$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix} = \sum \bar{i}\left[\frac{\partial}{\partial y}[zf(r)] - \frac{\partial}{\partial z}[yf(r)]\right]$$

$$\sum \bar{i}\left[zf^1(r)\frac{\partial r}{\partial y} - yf^1(r)\frac{\partial r}{\partial z}\right] = \sum \bar{i}\left[zf^1(r)\frac{y}{r} - yf^1(r)\frac{z}{r}\right]$$

= $\bar{0}$.

Example 9: If $\bar{A}$ is Irrotational vector, evaluate div($\bar{A}$ x $\bar{r}$) where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$.

Sol: we have $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Given $\bar{A}$ is an irrational vector

∇x$\bar{A}$ = $\bar{0}$

div ($\bar{A}$ x $\bar{r}$) = ∇.($\bar{A}$ x $\bar{r}$)

= $\bar{r}$.(∇x$\bar{A}$)-$\bar{A}$.(∇x$\bar{r}$)

= $\bar{r}$.($\bar{0}$)-$\bar{A}$.(∇x$\bar{r}$) [using (1)]

= -$\bar{A}$.(∇x$\bar{r}$).....(2)

Now $\nabla \text{x} \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} =$

$$\bar{i}\left(\frac{\partial}{\partial y}z - \frac{\partial}{\partial z}y\right) - \bar{j}\left(\frac{\partial}{\partial x}z - \frac{\partial}{\partial z}x\right) + \bar{k}\left(\frac{\partial}{\partial x}y - \frac{\partial}{\partial y}x\right) = \bar{0}$$

$\bar{A}.(\nabla \text{x} \bar{r})=0$ …(3)

Hence div ($\bar{A} \text{x} \bar{r}$)=0. [using (2) and (3)]

Example 10: Find constants a,b,c so that the vector $\bar{A}=$ $(x+2y+az)\bar{i}+(bx-3y-z)\,\bar{j}+(4x+cy+2z)\,\bar{k}$ is Irrotational. Also find ϕ such that $\bar{A}=\nabla\phi$.

Sol: Given vector is $\bar{A}=(x+2y+az)\bar{i}+(bx-3y-z)\,\bar{j}+(4x+cy+2z)\,\bar{k}$

Vector $\bar{A}$ is Irrotational $\Rightarrow$ curl $\bar{A}$ = $\bar{0}$

$$\Rightarrow \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \bar{0}$$

$\Rightarrow (c+1)\bar{i}+(a-4)\,\bar{j}+(b-2)\,\bar{k}=\bar{0}$

$\Rightarrow (c+1)\bar{i}+(a-4)\,\bar{j}+(b-2)\,\bar{k} = 0\bar{i}+0\,\bar{j}+0\,\bar{k}$

Comparing both sides,

c+1=0, a-4=0, b-2=0

c= -1, a=4,b=2

now $\bar{A}=(x+2y+az)\bar{i}+(bx-3y-z)\bar{j}+(4x+cy+2z)\bar{k}$, on substituting the values of a,b,c

we have $\bar{A}=\nabla\phi$.

$\Rightarrow \bar{A}=(x+2y+az)\bar{i}+(bx-3y-z)\bar{j}+(4x+cy+2z)\bar{k}= \bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}$

Comparing both sides, we have

$\frac{\partial\phi}{\partial x}=x+2y+4z \Rightarrow \phi= x^2/2+2xy+4zx+f_1(y,z)$

$\frac{\partial\phi}{\partial y}=2x-3y-z \Rightarrow \phi= 2xy-3y2/2-yz+f_2(z,x)$

$\frac{\partial\phi}{\partial z}=4x-y+2z \Rightarrow \phi= 4xz-yz+z2+f_3(x,y)$

Hence $\phi= x^2/2 -3y^2/2+z^2+2xy+4zx-yz+c$

Example 11: If ω is a constant vector, evaluate curl V where V = ωx$\bar{r}$.

Sol: curl (ωx$\bar{r}$) = $\sum \bar{i}x\frac{\partial}{\partial x}(\varpi x\bar{r}) = \sum \bar{i}x\left[\frac{\partial\varpi}{\partial x}x\bar{r}+\varpi x\frac{\partial\bar{r}}{\partial x}\right]$

$$=\sum \bar{i}x[\bar{0}+\omega x\bar{i}] \quad [\therefore \bar{a}x(\bar{b}x\bar{c})=(\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b}).\bar{c}]$$

$$=\sum \bar{i}x(\omega x\bar{i})=\sum[(\bar{i}.\bar{i})\omega-(\bar{i}.\omega)\bar{i}]=\sum\omega-\sum(\bar{i}.\omega)\bar{i}=3\omega-\omega=2\omega$$

Assignments

1. If $\bar{f}$ = $e^{x+y+z}(\bar{i}+\bar{j}+\bar{k})$ find curl $\bar{f}$.

2. Prove that $\bar{f} = (y+z)\bar{i}+(z+x)\bar{j}+(x+y)\bar{k}$ is Irrotational.

3. Prove that $\nabla.(\bar{a}x\,\bar{f}) = -\bar{a}$. curl $\bar{f}$ where $\bar{a}$ is a constant vector.

4. Prove that curl $(\bar{a}x\,\bar{r}) = 2\bar{a}$ where $\bar{a}$ is a constant vector.

5. if $\bar{f} = x^2y\bar{i}-2zx\,\bar{j}+2yz\,\bar{k}$ find (i) curl $\bar{f}$ (ii) curl curl $\bar{f}$.

OPERATORS

Vector differential operator ∇

The operator $\nabla = \bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}$ is defined such that $\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}$ where ϕ is a scalar point function.

Note: If ϕ is a scalar point function then $\nabla\phi$= grad $\phi = \sum i\frac{\partial\phi}{\partial x}$

(2) Scalar differential operator $\bar{a}.\nabla$

The operator $\bar{a}.\nabla = (\bar{a}.\bar{i})\frac{\partial\phi}{\partial x}+(\bar{a}.\bar{j})\frac{\partial\phi}{\partial y}+(\bar{a}.\bar{k})\frac{\partial\phi}{\partial z}$ is defined such that

$(\bar{a}.\nabla)\phi = (\bar{a}.\bar{i})\frac{\partial\phi}{\partial x}+(\bar{a}.\bar{j})\frac{\partial\phi}{\partial y}+(\bar{a}.\bar{k})\frac{\partial\phi}{\partial z}$

And $(\bar{a}.\nabla)\bar{f} = (\bar{a}.\bar{i})\frac{\partial\bar{f}}{\partial x}+(\bar{a}.\bar{j})\frac{\partial\bar{f}}{\partial y}+(\bar{a}.\bar{k})\frac{\partial\bar{f}}{\partial z}$

(3). Vector differential operator $\bar{a}x\nabla$

The operator $\bar{a}x\nabla = (\bar{a}x\bar{i})\frac{\partial}{\partial x}+(\bar{a}x\bar{j})\frac{\partial}{\partial y}+(\bar{a}x\bar{k})\frac{\partial}{\partial z}$ is defined such that

(i). $(\bar{a}x\nabla)\phi = (\bar{a}x\bar{i})\frac{\partial\phi}{\partial x}+(\bar{a}x\bar{j})\frac{\partial\phi}{\partial y}+(\bar{a}x\bar{k})\frac{\partial\phi}{\partial z}$

(ii). $(\bar{a} \times \nabla).\bar{f} = (\bar{a} x \bar{i}).\frac{\partial \bar{f}}{\partial x} + (\bar{a} x \bar{j}).\frac{\partial \bar{f}}{\partial y} + (\bar{a} x \bar{k}).\frac{\partial \bar{f}}{\partial z}$

(iii). $(\bar{a} \times \nabla) x \bar{f} = (\bar{a} x \bar{i}) x \frac{\partial \bar{f}}{\partial x} + (\bar{a} x \bar{j}) x \frac{\partial \bar{f}}{\partial y} + (\bar{a} x \bar{k}) x \frac{\partial \bar{f}}{\partial z}$

(4). Scalar differential operator ∇.

The operator $\nabla = \bar{i}.\frac{\partial}{\partial x} + \bar{j}.\frac{\partial}{\partial y} + \bar{k}.\frac{\partial}{\partial z}$ is defined such that

$\nabla.\bar{f} = \bar{i}.\frac{\partial \bar{f}}{\partial x} + \bar{j}.\frac{\partial \bar{f}}{\partial y} + \bar{k}.\frac{\partial \bar{f}}{\partial z}$

Note: $\nabla.\bar{f}$ is defined as div $\bar{f}$ it is a scalar point function.

(5). Vector differential operator ∇ x

The operator $\nabla x = \bar{i} x \frac{\partial}{\partial x} + \bar{j} x \frac{\partial}{\partial y} + \bar{k} x \frac{\partial}{\partial z}$ is defined such that

$$\nabla x \bar{f} = \bar{i} x \frac{\partial \bar{f}}{\partial x} + \bar{j} x \frac{\partial \bar{f}}{\partial y} + \bar{k} x \frac{\partial \bar{f}}{\partial z}$$

Note : $\nabla x \bar{f}$ is defined as curl $\bar{f}$. It is a vector point function.

(6). Laplacian Operator ∇^2

$$\nabla.\nabla\phi = \sum \bar{i}.\frac{\partial}{\partial x}\left(\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}\right) = \sum \frac{\partial^2\phi}{\partial x^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$$

Thus the operator $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called Laplacian operator.

Note : (i). $\nabla^2\phi = \nabla.(\nabla\phi)$ = div(grad ϕ)

(ii). if $\nabla^2\phi = 0$ then ϕ is said to satisfy Laplacian equation. This ϕ is called a harmonic function.

SOLVED EXAMPLES

Example 1: Prove that div.(grad r^m)= $m(m+1)r^{m-2}$ (or) $\nabla^2(r^m) = m(m+1)r^{m-2}$ (or) $\nabla^2(r^n) = n(n+1)r^{n-2}$

Sol: Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ and r = $|\bar{r}|$ then $r^2 = x^2+y^2+z^2$.

Differentiating w.r.t. 'x' partially, wet get $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$.

Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

Now grad(r^m) = $\sum \bar{i}\frac{\partial}{\partial x}(r^m) = \sum \bar{i} m r^{m-1}\frac{\partial r}{\partial x} = \sum \bar{i} m r^{m-1}\frac{x}{r} = \sum \bar{i} m r^{m-2} x$

div (grad rm) = $\sum \bar{i}\frac{\partial}{\partial x}[mr^{m-2}x] = m\sum\left[(m-2)r^{m-3}\frac{\partial r}{\partial x}x + r^{m-2}\right]$

$= m\sum\left[(m-2)r^{m-4}x^2 + r^{m-2}\right] = m\left[(m-2)r^{m-4}\sum x^2 + \sum r^{m-2}\right]$

$= m[(m-2)r^{m-4}(r^2)+3r^{m-2}]$

$= m[(m-2)\, r^{m-2}+3r^{m-2}] = m[(m-2+3)r^{m-2}] = m(m+1)r^{m-2}$.

Hence $\nabla^2(r^m) = m(m+1)r^{m-2}$

Example 2: Show that $\nabla^2[f(r)] = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} = f^{11}(r) + \frac{2}{r}f^1(r)$ where r = $|\bar{r}|$.

Sol: grad [f(r)] = $\nabla f(r) = \sum i\frac{\partial}{\partial x}[f(r)] = \sum i f^1(r)\frac{\partial r}{\partial x} = \sum i f^1(r)\frac{x}{r}$

div [grad f(r)] = $\nabla^2[f(r)] = \nabla.\nabla f(r) = \sum \frac{\partial r}{\partial x}\left[f^1(r)\frac{x}{r}\right]$

$$= \sum \frac{r\frac{\partial}{\partial x}[f^1(r)x] - f^1(r)x\frac{\partial}{\partial x}(r)}{r^2}$$

$$= \sum \frac{r\left(f^{11}(r)\frac{\partial r}{\partial x}x + f^1(r)\right) - f^1(r)x\left(\frac{x}{r}\right)}{r^2}$$

$$= \sum \frac{rf^{11}(r)\frac{x}{r}x + rf^1(r) - f^1(r)x\left(\frac{x}{r}\right)}{r^2}$$

$$= \frac{\sum rf^{11}(r).\frac{x}{r}.x + f^1(r) - x^2.}{r^2}.\frac{f^1(r)}{r}$$

$$= \frac{f^{11}(r)}{r^2}\sum x^2 + \frac{1}{r}\sum f^1(r) - \frac{1}{r^3}f^1(r)\sum x^2$$

$$= \sum{}' \frac{f^{11}(r)}{r^2}(r^2) + \frac{3}{r}f^1(r) - \frac{1}{r^3}f^1(r).r^2$$

$$= f^{111}(r) + \frac{2}{r}f^1(r)$$

Example 3: If ϕ satisfies Laplacian equation, show that $\nabla\phi$ is both solenoidal and Irrotational.

Sol: given $\nabla^2\phi = 0 \Rightarrow$div(grad ϕ)= 0 $\Rightarrow$ grad ϕ is solenoidal

We know that curl (grad ϕ) = $\bar{0} \Rightarrow$grad ϕ is always Irrotational.

Example 4: Show that (i) $(\bar{a}.\nabla)\phi = \bar{a}.\nabla\phi$ (ii) $(\bar{a}.\nabla)\bar{r} = \bar{a}$.

Sol: (i). Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$. Then

$$\bar{a}.\nabla = (a_1\bar{i} + a_2\bar{j} + a_3\bar{k}).(\bar{i}.\frac{\partial}{\partial x} + \bar{j}.\frac{\partial}{\partial y} + \bar{k}.\frac{\partial}{\partial z}) = a_1.\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}$$

$$(\bar{a}.\nabla)\phi = a_1\frac{\partial\phi}{\partial x} + a_2\frac{\partial\phi}{\partial y} + a_3\frac{\partial\phi}{\partial z}$$

Hence $(\bar{a}.\nabla)\phi = \bar{a}.\nabla\phi$

(ii). $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\therefore \frac{\partial\bar{r}}{\partial x} = \bar{i} \quad \frac{\partial\bar{r}}{\partial y} = \bar{j} \quad \frac{\partial\bar{r}}{\partial z} = \bar{k}$$

$$(\bar{a}.\nabla)\bar{r} = \sum a_1\frac{\partial}{\partial x}(\bar{r}) = \sum a_1\frac{\partial}{\partial x} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k} = \bar{a}$$

Example 5: Prove that (i) $(\bar{f}\text{x}\nabla)\bar{r} = 0$ (ii). $(\bar{f}\text{x}\nabla)\text{x}\bar{r} = -2\bar{f}$

Sol: (i) $(\bar{f}\text{x}\nabla)\bar{r} = \sum(\bar{f}x\bar{i}).\frac{\partial\bar{r}}{\partial x} = \sum(\bar{f}x\bar{i}).\bar{i} = 0$

(ii) $(\bar{f}\text{x}\nabla) = (\bar{f}x\bar{i})\frac{\partial}{\partial x}x(\bar{f}x\bar{j})\frac{\partial}{\partial y}x(\bar{f}x\bar{k})\frac{\partial}{\partial z}$

$$(\bar{f}\text{x}\nabla)\text{x}\bar{r} = (\bar{f}x\bar{i})x\frac{\partial\bar{r}}{\partial x} + (\bar{f}x\bar{j})x\frac{\partial\bar{r}}{\partial y} + (\bar{f}x\bar{k})x\frac{\partial\bar{r}}{\partial z} = \sum(\bar{f}x\bar{i})x\bar{i} = \sum\left[(\bar{f}.i)i - \bar{f}\right]$$

$$= (\bar{f}.i)\bar{i} + (\bar{f}.\bar{j})\bar{j} + (\bar{f}.\bar{k})\bar{k} - 3\bar{f} = -2\bar{f}.$$

Example 6: Find div $\bar{F}$. Where $\bar{F}$ = grad $(x^3+y^3+z^3-3xyz)$

Sol: Let $\phi = x^3+y^3+z^3-3xyz$. Then

$\bar{F}$ = grad ϕ

$$= \sum i\frac{\partial\phi}{\partial x} = 3(x^2 - yz)\bar{i} + 3(y^2 - zx)\bar{j} + 3(x^2 - xy)\bar{k} = F_1i + F_2j + F_3k \ (say)$$

div $\bar{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$ = 6x+6y+6z= 6(*x*+*y*+*z*)

i.e div[grad($x^3+y^3+z^3$-3xyz)]= $\nabla^2(x^3+y^3+z^3-3xyz)$= 6(*x*+*y*+*z*).

Example 7: If $f = (x^2+y^2+z^2)^{-n}$ then find div grad *f* and determine n if div grad *f*= 0.

Sol: let $f = (x^2+y^2+z^2)^{-n}$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$r = |\bar{r}| \Rightarrow r^2 = x^2+y^2+z^2$

$\Rightarrow f(r) = (r^2)^{-n} = r^{-2n}$

$f^1(r) = -2n\, r^{-2n-1}$

and $f^{11}(r) = (-2n)(-2n-1)r^{-2n-2} = 2n(2n+1)r^{-2n-2}$

We have div grad f = $\nabla^2 f(r) = f^{11}(r) + {}^2/_r f^1(r) = (2n)(2n+1)r^{-2n-2} - 4n\, r^{-2n-2}$

$= r^{-2n-2}[2n(2n+1-2)] = (2n)(2n-1)r^{-2n-2}$

If div grad *f*(r) is zero, we get n = 0 or n = ½ .

Example 8: Prove that $\nabla x\left(\frac{\bar{A}x\bar{r}}{r^n}\right)=\frac{(2-n)\bar{A}}{r^n}+\frac{n(\bar{r}.\bar{A})\bar{r}}{r^{n+2}}$.

Sol: we have $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ and $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$\therefore \frac{\partial\bar{r}}{\partial x}=\bar{i}\ \frac{\partial\bar{r}}{\partial y}=\bar{j}\ \frac{\partial\bar{r}}{\partial z}=\bar{k}$ and

$r^2 = x^2+y^2+z^2$…..(1)

Diff. (1) partially,

$2r\frac{\partial r}{\partial x}=2x\Rightarrow\frac{\partial r}{\partial x}=\frac{x}{r}$, similarly $\frac{\partial r}{\partial y}=\frac{y}{r}, and\ \frac{\partial r}{\partial z}=\frac{z}{r}$

$$\nabla x\left(\frac{\bar{A}x\bar{r}}{r^n}\right)=\sum\bar{i}x\frac{\partial}{\partial x}\left(\frac{(\bar{A}x\bar{r})}{r^n}\right)$$

Now $\frac{\partial}{\partial x}\left(\frac{(\bar{A}x\bar{r})}{r^n}\right)=\bar{A}x\frac{\partial}{\partial x}\left(\frac{\bar{r}}{r^n}\right)=\bar{A}x\left[\frac{r^n\bar{i}-\bar{r}nr^{n-1}}{r^{2n}}\right]\frac{\partial r}{\partial x}$

$$=\bar{A}x\left[\frac{r^n\bar{i}-nr^{n-2}x\bar{r}}{r^{2n}}\right]=\bar{A}x\left[\frac{1}{r^n}\bar{i}-\frac{n}{r^{n+2}}.x\bar{r}\right]$$

$$=\frac{\bar{A}x\bar{i}}{r^n}-\frac{n}{r^{n+2}}.x(\bar{A}x\bar{r})$$

$$\bar{i}x\frac{\partial}{\partial x}\left(\frac{(\bar{A}x\bar{r})}{r^n}\right)=\frac{\bar{i}x(\bar{A}x\bar{i})}{r^n}-\frac{nx}{r^{n+2}}.\bar{i}x(\bar{A}x\bar{r})$$

$$=\frac{(\bar{i}.\bar{i})\bar{A}-(\bar{i}.\bar{A})\bar{i}}{r^n}=\frac{nx}{r^{n+2}}[((\bar{i}.\bar{r})\bar{A}-(\bar{i}.\bar{A})\bar{r})]$$

Let $A_1\bar{i}+A_2\bar{j}+A_3\bar{k}$. Then $\bar{i}.\bar{A}=A_1$

$$\bar{i}x\frac{\partial}{\partial x}\left(\frac{(\bar{A}x\bar{r})}{r^n}\right)=\left(\frac{\bar{A}-A_1\bar{i}}{r^n}\right)-\frac{nx}{r^{n+2}}[r\bar{A}-A_1\bar{r}]$$

And $\Sigma\bar{i}x\frac{\partial}{\partial x}\left(\frac{(\bar{A}x\bar{r})}{r^n}\right)=\Sigma\left(\frac{\bar{A}-A_1\bar{i}}{r^n}\right)-\frac{nx}{r^{n+2}}[r\bar{A}-A_1\bar{r}]$

$$= \frac{3\overline{A} - \overline{A}}{r^n} - \frac{n}{r^{n+2}}[r^2\overline{A}] + \frac{n\overline{r}}{r^{n+2}}(A_1x + A_2y + A_3z)$$

$$\frac{2\overline{A}}{r^n} - \frac{n}{r^n}\overline{A} + \frac{n\overline{r}}{r^{n+2}}(\overline{A}.\overline{r}) = \frac{(2-n)\overline{A}}{r^n} + \frac{n\overline{r}}{r^{n+2}}(\overline{A}.\overline{r})$$

Hence the result.

VECTOR IDENTITIES

Theorem 1: If $\overline{a}$ is a differentiable function and ϕ is a differentiable scalar function. Then prove that div(ϕ$\overline{a}$)= (grad ϕ).$\overline{a}$+ϕ div $\overline{a}$ or ∇.(ϕ$\overline{a}$)= (∇ϕ)$\overline{a}$+ϕ(∇.$\overline{a}$)

Proof: div(ϕ$\overline{a}$)=∇.(ϕ$\overline{a}$)=$\sum i\frac{\partial}{\partial x}(\phi\overline{a})$

$$=\sum \overline{i}.\left(\frac{\partial\phi}{\partial x}\overline{a} + \phi\frac{\partial\overline{a}}{\partial x}\right) = \sum\left(i\frac{\partial\phi}{\partial x}\overline{a}\right) + \sum\left(i\frac{\partial\overline{a}}{\partial x}\right)\phi$$

$$= \sum\left(\overline{i}\frac{\partial\phi}{\partial x}\right).\overline{a} + \left(\sum\overline{i}\frac{\partial a}{\partial x}\right)\phi = (\nabla\phi)\overline{a} + \phi(\nabla.\overline{a})$$

Theorem 2: prove that curl (ϕ$\overline{a}$)= (grad ϕ)x$\overline{a}$+ϕ curl $\overline{a}$

Proof : curl (ϕ$\overline{a}$)=∇x(ϕ$\overline{a}$)=$\sum ix\frac{\partial}{\partial x}(\phi\overline{a})$

$$= \sum\overline{i}x\left(\frac{\partial\phi}{\partial x}\overline{a} + \phi\frac{\partial\overline{a}}{\partial x}\right) = \sum\left(i\frac{\partial\phi}{\partial x}\right)x\overline{a} + \sum\left(ix\frac{\partial\overline{a}}{\partial x}\right)\phi$$

$$= \nabla\phi x\overline{a} + (\nabla x\overline{a})\phi = (\text{grad }\phi)x\overline{a} + \phi \text{ curl } \overline{a}$$

Theorem 3: Prove that grad ($\bar{a}.\bar{b}$)=

$(\bar{b}.\nabla)\bar{a} + (\bar{a}.\nabla)\bar{b} + \bar{b}x\, curl\, \bar{a} + \bar{a}x\, curl\, \bar{b}$

Proof: Consider

$$\bar{a}x\, curl(\bar{b}) = \bar{a}x(\nabla x\bar{b}) = \bar{a}x\sum \bar{i}x\left(\bar{i}x\frac{\partial \bar{b}}{\partial x}\right)$$

$$= \sum \bar{a}x\left(\bar{i}x\frac{\partial \bar{b}}{\partial x}\right)$$

$$= \sum\left\{\left(\bar{a}.\frac{\partial \bar{b}}{\partial x}\right)\bar{i} - (\bar{a}.\bar{i})\frac{\partial \bar{b}}{\partial x}\right\} = \sum \bar{i}\left\{\bar{a}.\frac{\partial \bar{b}}{\partial x}\right\} - \left\{\bar{a}.\sum i\frac{\partial}{\partial x}\right\}\bar{b}$$

$$\therefore \bar{a}x\, curl\, \bar{b} = \sum \bar{i}\left(\bar{a}.\frac{\partial \bar{b}}{\partial x}\right) - (\bar{a}.\nabla)\bar{b} \quad ....(1)$$

Similarly , $\bar{b}x\, curl\, \bar{b} = \sum \bar{i}\left(\bar{a}.\frac{\partial \bar{a}}{\partial x}\right) - (\bar{b}.\nabla)\bar{a}$ ....(2)

(1)+(2) gives

$$\bar{a}x\, curl\, \bar{b} + \bar{b}xcurl\, \bar{a} = \sum \bar{i}\left(\bar{a}.\frac{\partial \bar{b}}{\partial x}\right) - (\bar{a}.\nabla)\bar{b} + \sum \bar{i}\left(\bar{b}.\frac{\partial \bar{a}}{\partial x}\right) - (\bar{b}.\nabla)\bar{a}$$

$$\bar{a}x\, curl\, \bar{b} + \bar{b}xcurl\, \bar{a} + (\bar{a}.\nabla)\bar{b} + (\bar{b}.\nabla)\bar{a} = \sum \bar{i}\left(\bar{a}.\frac{\partial \bar{b}}{\partial x} + \bar{b}.\frac{\partial \bar{a}}{\partial x}\right)$$

$$= \sum \bar{i}\frac{\partial}{\partial x}(\bar{a}.\bar{b})$$

$$= \nabla(\bar{a}.\bar{b}) = \text{grad}\ (\bar{a}.\bar{b})$$

Theorem 4: Prove that div ($\bar{a}\,\mathrm{x}\,\bar{b}$)= $\bar{b}.curl\, \bar{a} - \bar{a}.curl\, \bar{b}$

Proof: div ($\bar{a}$ x $\bar{b}$) $=\sum \bar{i}\,\frac{\partial}{\partial x}(\bar{a}x\bar{b})=\sum \bar{i}.\left(\frac{\partial \bar{a}}{\partial x}x\bar{b}+\bar{a}x\frac{\partial \bar{b}}{\partial x}\right)$

$$=\sum \bar{i}.\left(\frac{\partial \bar{a}}{\partial x}x\bar{b}\right)+\sum \bar{i}.\left(\bar{a}x\frac{\partial \bar{b}}{\partial x}\right)=\sum\left(\bar{i}x\frac{\partial \bar{a}}{\partial x}\right).\bar{b}-\sum\left(\bar{i}x\frac{\partial \bar{b}}{\partial x}\right)\bar{a}$$

$$=(\nabla x\bar{a}).\bar{b}-(\nabla x\bar{b}).\bar{a}=\bar{b}.curl\,\bar{a}-\bar{a}.curl\,\bar{b}$$

Theorem 5 : $curl\,(\bar{a}x\bar{b})=\bar{a}\,div\,\bar{b}-\bar{b}\,div\,\bar{a}+(\bar{b}.\nabla)\bar{a}-(\bar{a}.\nabla)\bar{b}$

$Proof:\ curl\,(\bar{a}x\bar{b})=\sum \bar{i}x\frac{\partial}{\partial x}(\bar{a}x\bar{b})=\sum \bar{i}x\left[\frac{\partial \bar{a}}{\partial x}x\bar{b}+\bar{a}x\frac{\partial \bar{b}}{\partial x}\right]$

$$\sum \bar{i}x\left(\frac{\partial \bar{a}}{\partial x}x\bar{b}\right)+\sum \bar{i}x\left(\bar{a}x\frac{\partial \bar{b}}{\partial x}\right)$$

$$=\sum\left\{(\bar{i}.\bar{b})\frac{\partial \bar{a}}{\partial x}-\left(\bar{i}.\frac{\partial \bar{a}}{\partial x}\right)\bar{b}\right\}+\sum\left\{\left(\bar{i}.\frac{\partial \bar{b}}{\partial x}\right)\bar{a}-(\bar{i}.\bar{a})\frac{\partial \bar{b}}{\partial x}-\right\}+$$

$$=\sum(\bar{b}.\bar{i})\frac{\partial \bar{a}}{\partial x}-\sum\left(\bar{i}.\frac{\partial \bar{a}}{\partial x}\right)\bar{b}+\sum\left(\bar{i}.\frac{\partial \bar{b}}{\partial x}\right)\bar{a}-\left(.\bar{a}\sum \bar{i}.\frac{\partial}{\partial x}\right)\bar{b}$$

$$=(\bar{b}.\nabla)\bar{a}-(\bar{a}.\nabla)\bar{b}+(\nabla.\bar{b})\bar{a}-(\bar{a}.\nabla)\bar{b}$$

$$=(\nabla.\bar{b})\bar{a}-(\nabla.\bar{a})\bar{b}+(\bar{b}.\nabla)\bar{a}-(\bar{a}.\nabla)\bar{b}$$

$$=\bar{a}\,div\,\bar{b}-\bar{b}\,div\,\bar{a}+(\bar{b}.\nabla)\bar{a}-(\bar{a}.\nabla)\bar{b}$$

Theorem 6: Prove that curl grad $\phi = 0$.

Proof: Let ϕ be any scalar point function. Then

$$.grad\ \phi=\bar{i}\frac{\partial \phi}{\partial x}+\bar{j}\frac{\partial \phi}{\partial y}+\bar{k}\frac{\partial \phi}{\partial z}$$

$$curl\ (\text{grad}\ \Phi) = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \frac{d\phi}{dx} & \frac{d\phi}{dy} & \frac{d\phi}{dz} \end{vmatrix}$$

$$\bar{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) - \bar{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z\partial x}\right) - \bar{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right) = \bar{0}$$

note: Since crul (grad ϕ)= 0, we have grad ϕ is always Irrotational.

Theorem 7: Prove that div crul *f=0*

$\text{Proof} : Let\ \bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$$\therefore curl\ \bar{f}\ \nabla x\bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\bar{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_2}{\partial z}\right)\bar{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\bar{k}$$

$$\therefore \quad div\ \ curl\ \bar{f} = \nabla.(\nabla x\bar{f}) = \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x} - \frac{\partial f_2}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$=\frac{\partial^2 f_3}{\partial x\partial y}-\frac{\partial^2 f_2}{\partial x\partial z}-\frac{\partial^2 f_3}{\partial y\partial x}+\frac{\partial^2 f_1}{\partial y\partial z}+\frac{\partial^2 f_2}{\partial z\partial x}-\frac{\partial^2 f_1}{\partial z\partial y}=0$$

Theorem 8: If f and g are two scalar point functions, prove that $div(f\nabla g)= f\nabla^2 g+\nabla f.\nabla g$

Sol: Let f and g are two scalar point functions. Then

$$\nabla g=\bar{i}\frac{\partial g}{\partial x}+\bar{j}\frac{\partial g}{\partial y}+\bar{k}\frac{\partial g}{\partial z}$$

Now $f\nabla g=\bar{i}f\frac{\partial g}{\partial x}+\bar{j}f\frac{\partial g}{\partial y}+\bar{k}f\frac{\partial g}{\partial z}$

$$\nabla.(f\nabla g)=\frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial x}\right)+\frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial y}\right)+\frac{\partial}{\partial z}\left(f\frac{\partial g}{\partial z}\right)$$

$$=f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial x^2}\right)+\left(\frac{\partial f}{\partial x}.\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}.\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}.\frac{\partial g}{\partial z}\right)$$

$$=f\nabla^2 g+\left(\bar{i}\frac{\partial f}{\partial x}+\bar{j}\frac{\partial f}{\partial y}+\bar{k}\frac{\partial f}{\partial z}\right).\left(\bar{i}\frac{\partial g}{\partial x}+\bar{j}\frac{\partial g}{\partial y}+\bar{k}\frac{\partial g}{\partial z}\right).$$

$$=f\nabla^2 g+\nabla f.\nabla g$$

Theorem 9: Prove that $\nabla x(\nabla x\bar{a})=\nabla(\nabla.\bar{a})-\nabla^2\bar{a}$.

Proof: $\nabla x(\nabla x\bar{a})=\sum\bar{i}\frac{\partial}{\partial x}(\nabla x\bar{a})$

Now $\bar{i}x\frac{\partial}{\partial x}(\nabla x\bar{a})=\bar{i}x\frac{\partial}{\partial x}\left(\bar{i}x\frac{\partial\bar{a}}{\partial x}+\bar{j}x\frac{\partial\bar{a}}{\partial y}+\bar{k}x\frac{\partial\bar{a}}{\partial z}\right)$

$$=\bar{i}x\left(\bar{i}x\frac{\partial^2\bar{a}}{\partial x^2}+\bar{j}x\frac{\partial^2\bar{a}}{\partial y^2}+\bar{k}x\frac{\partial^2\bar{a}}{\partial z^2}\right)$$

$$= \bar{i}x\left(\bar{i}x\frac{\partial^2\bar{a}}{\partial x^2}\right) + \bar{i}x\left(\bar{j}x\frac{\partial^2\bar{a}}{\partial x\partial y}\right) + \bar{i}x\left(\bar{k}x\frac{\partial^2\bar{a}}{\partial x\partial z}\right)$$

$$= \left(\bar{i}.\frac{\partial^2\bar{a}}{\partial x^2}\right)\bar{i} - \frac{\partial^2\bar{a}}{\partial x^2} + \left(\bar{i}.\frac{\partial^2\bar{a}}{\partial x\partial y}\right)\bar{j} + \left(\bar{i}.\frac{\partial^2\bar{a}}{\partial x\partial z}\right)\bar{k} \quad [\because i.i = 1, i.j = i.k = 0]$$

$$= \bar{i}\frac{\partial}{\partial x}\left(\bar{i}.\frac{\partial\bar{a}}{\partial x}\right) + j\frac{\partial}{\partial y}\left(\bar{i}.\frac{\partial\bar{a}}{\partial y}\right) + k\frac{\partial}{\partial z}\left(\bar{i}.\frac{\partial\bar{a}}{\partial z}\right) - \frac{\partial^2\bar{a}}{\partial x^2} = \nabla\left(\bar{i}.\frac{\partial\bar{a}}{\partial x}\right) - \frac{\partial^2\bar{a}}{\partial x^2}$$

$$= \sum\bar{i}x\frac{\partial}{\partial x}(\nabla x\bar{a}) = \sum\bar{i}.\frac{\partial\bar{a}}{\partial x} - \sum\frac{\partial^2\bar{a}}{\partial x^2} = \nabla(\nabla.\bar{a}) - \left(\frac{\partial^2\bar{a}}{\partial x^2} + \frac{\partial^2\bar{a}}{\partial y^2} + \frac{\partial^2\bar{a}}{\partial z^2}\right)$$

$$= \nabla x(\nabla x\bar{a}) = \nabla(\nabla.\bar{a}) - \nabla^2\bar{a}$$

SOLVED EXAMPLES

Example 1: Prove that $(\nabla f \, x \nabla g)$is solenoidal.

Sol: We know that div ($\bar{a} \, x \, \bar{b}$) = $\bar{b}.curl\,\bar{a} - \bar{a}.curl\,\bar{b}$ (see Theorem 4)

Take $\bar{a} = \bar{\nabla}f$ and b= ∇g

Then div (∇f x ∇g) = ∇g. curl (∇f) - ∇f. curl (∇g)=0

$\nabla f \, x \nabla g$ is solenoidal.

Example 2: Prove that (i) div. $\{(\bar{r} \times \bar{a}) \times \bar{b}\} = -2(\bar{b}.\bar{a})$

(ii) curl {(r. a) x b=b x a where a and b are constant vectors.

Sol: (i) .

$$div\{(\bar{r} x \bar{a}) x \bar{b}\} = div[(\bar{r}.\bar{b})\bar{a} - (\bar{a}..\bar{b})\bar{r}]$$

$$= div(\bar{r}.\bar{b})\bar{a} - (\bar{a}..\bar{b})\bar{r}$$

$$[(\bar{r}.\bar{b})div\,\bar{a} + \bar{a}.grad(\bar{r}.\bar{b})] - [(\bar{a}.\bar{b})div\,\bar{r} + \bar{r}.grad(\bar{a}.\bar{b})]$$

We have $div\ \bar{a} = 0, div\,\bar{r} = 3, grad\ (\bar{a}.\bar{b}) = 0$

$$div\ \{(\bar{r}X\bar{a})X\bar{b}\} = 0 + \bar{a}\ grad(\bar{r}.\bar{b}) - 3(\bar{a}.\bar{b})$$

$$= \bar{a}\sum \frac{id}{dx}(\bar{r}.\bar{b}) - 3(\bar{a}.\bar{b})$$

$$= \bar{a}\sum \bar{\imath}\frac{d\bar{r}}{dx}.\bar{b} - 3(\bar{a}.\bar{b})$$

$$= \bar{a}\sum \bar{\imath}\,(\bar{\imath}.\bar{b}) - 3(\bar{a}.\bar{b})$$

$$= \bar{a}.b - 3(\bar{a}.\bar{b}) = -2(\bar{a}.\bar{b})$$

$$= -2(\bar{b}.\bar{a})$$

Example 3: Prove that $\nabla\left[\nabla.\frac{\bar{r}}{r}\right] = \frac{-2}{r^3}\bar{r}.$

Sol: we have $\nabla.\left(\frac{\bar{r}}{r}\right) = \sum i.\frac{\partial}{\partial x}\left(\frac{\bar{r}}{r}\right)$

$$= \sum i\left[\frac{1}{r}\frac{\partial \bar{r}}{\partial x} + \bar{r}\left(\frac{-1}{r^2}\right)\left(\frac{x}{r}\right)\right] = \sum i.\left(\frac{1}{r}i - \frac{\bar{r}}{r^3}x\right)$$

$$= \frac{1}{r}\sum i.i - \frac{1}{r^3}r^2 = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

$$\nabla\left[\nabla.\left(\frac{\bar{r}}{r}\right)\right]=\sum i.\left(\frac{\partial}{\partial x}\left(\frac{2}{r}\right)\right)=\sum i.\left(\frac{-2}{r^2}\right)\left(\frac{x}{r}\right)=\frac{-2}{r^3}\sum xi=\frac{-2\bar{r}}{r^3}.$$

Example 4: Find (Ax∇)ϕ, if A = yz^2 i- $3xz^2$ j+2xyzk and ϕ = xyz.

Sol : We have

$$A x \nabla=\begin{vmatrix} i & j & k \\ yz^2 & -3xz^2 & 2xyz \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial x}(-3xz^2)-\frac{\partial}{\partial y}(2xyz)\right]-j\left[\frac{\partial}{\partial z}(yz^2)-\frac{\partial}{\partial x}(2xyz)\right]+k\left[\frac{\partial}{\partial y}(yz^2)-\frac{\partial}{\partial x}(-3xz^2)\right]$$

=(-6xz-2xz)-j(2yz-2yz)+k(z^2+$3z^2$)= -8*x*z i-0j+$4z^2$k

(Ax∇)ϕ, = (-8*x*z i+$4z^2$k)xyz = $-8x^2yz^2$ i+$4xyz^3$k

UNIT-V
Vector Integration

1. Line integral:- (i) $\int_c \bar{F}.d\bar{r}$ is called Line integral of $\bar{F}$ along c

Note : Work done by $\bar{F}$ along a curve c is $\int_c \bar{F}.d\bar{r}$

Example 5: If $\bar{F}(x^2-27)\,\bar{i}-6yz\,\bar{j}+8xz^2\,\bar{k}$, evaluate $\int \bar{F}.d\bar{r}$ from the point (0,0,0) to the point (1,1,1) along the Straight line from (0,0,0) to (1,0,0), (1,0,0) to (1,1,0) and (1,1,0) to (1,1,1).

Solution : Given $\bar{F}=(x^2-27)\bar{i}-6yz\bar{j}+8xz^2\bar{k}$

Now $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k} \Rightarrow d\bar{r}\ dx\bar{i}+dy\bar{j}+dz\bar{k}$

$\therefore\quad \bar{F}.d\bar{r}=(x^2-27)dx-(6yz)dy+8xz^2dz$

(i) Along the straight line from O = (0,0,0) to A = (1,0,0)

Here y =0 =z and dy=dz=0. Also x changes from 0 to 1.

$$\int_{OA} \bar{F}.d\bar{r}=\int_o^1 (x^2-27)dx=\left[\frac{x^3}{3}-27x\right]_0^1=\frac{1}{3}-27=\frac{-80}{3}$$

(ii) Along the straight line from A = (1,0,0) to B = (1,1,0)

Here x =1, z=0 $\Rightarrow$ dx=0, dz=0. y changes from 0 to 1.

$$\int_{AB} \bar{F}\,.\,d\bar{r} = \int_{y=0}^{1} (-6yz)dy = 0$$

(iii) Along the straight line from B = (1,1,0) to C = (1,1,1)

x =1 =y $\Rightarrow$ dx=dy=0 and z changes from 0 to 1.

$$\int_{BC} \bar{F}\,.\,d\bar{r} = \int_{z=0}^{1} 8xz^2 dz = \int_{z=0}^{1} 8xz^2 dz = \left[\frac{8z^3}{3} -\right]_0^1 = \frac{8}{3}$$

$$(i)+(ii)+(iii) \Rightarrow \int_C \bar{F}\,.\,d\bar{r} = \frac{88}{3}$$

<u>Example : 6</u> If $\bar{F}$(5xy-6x^2)$\bar{i}$ (2y-4x)$\bar{j}$, evaluate $\int_C \bar{F}.d\bar{r}$ along the curve C in xy plane y=x^{3}from (1,1) to (2,8).

Solution : Given $\bar{F}$(5xy-6x^2)$\bar{i}$ (2y-4x)$\bar{j}$,

Along the curve y=x^2, dy =3x^2 dx

$\therefore$ $\bar{F}$(5x^4-6x^2)$\bar{i}$+(2x^3-4x)$\bar{j}$, [Putting y=x^3 in (1)]

d$\bar{r}$= dx$\bar{i}$+dy$\bar{j}$+dx$\bar{i}$ +3x^{2}dx

$\therefore$ $\bar{F}$.d$\bar{r}$= [(5x^4-6x^2)$\bar{i}$+(2x^3-4x)$\bar{j}$].dx$\bar{i}$ +3x^{2}dx$\bar{j}$

= (5x^4 – 6x^2) dx+(2x^3 – 4x)3x^{3}dx

= (6x^5+5x^4-12x^3 -6x^2)dx

Hence $\int_{y=x^3} \bar{F}.d\bar{r} = \int_1^2 (6x^5+5x^4-12x^3-6x^2)dx$

$$= \left(6.\frac{x^6}{6} + 5.\frac{x^3}{5} - 12.\frac{x^4}{4} - 6.\frac{x^3}{4}\right) = \left(x^6 + x^3 - 3x^4 - 2x3\right)_1^2$$

= 16(4+2-31) – (1+1-3-2) = 32+3 = 35

Example 7: Find the work done by the force $\bar{F} = z\bar{i}+x\bar{j}+y\bar{k}$, when it moves a particle along the arc of the curve $\bar{r}$ = cost $\bar{i}$ + sint $\bar{j}$-t $\bar{k}$ from t = 0 to t = 2π

Solution : Given force $\bar{F}$ = zi = xj +yk and the arc is $\bar{r}$ = cost $\bar{i}$ + sin t $\bar{j}$-t

i.e., x = cost, y= sin t, z = -t

d$\bar{r}$= (-sin t $\bar{i}$ +cost $\bar{j}$-$\bar{k}$)dt

$\bar{F}$.d$\bar{r}$= (-t $\bar{i}$+cost $\bar{j}$+sin t $\bar{k}$). (-sin t $\bar{i}$ + cost $\bar{j}$- $\bar{k}$)dt = (t sin t + $\cos^2$ t – sin t)dt

Hence work done = $\int_0^{2\pi} \bar{F}.d\bar{r} = \int_0^{2\pi}(t \sin t + \cos^2 t - \sin t)\, dt$

$$= [t(-\cos t)]_0^{2\pi} - \int_0^{2\pi}(-\sin t)dt +)+ \int_0^{2\pi}\frac{1+\cos 2t}{2}dt - \int_0^{2\pi}\sin t\, dt$$

$$= -2\pi - (\cos t)_0^{2\pi} + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)_0^{2\pi} + (\cos t)_0^{2\pi}$$

$$= -2\pi - (1-1) + \frac{1}{2}(2\pi) + (1-1) = -2\pi + \pi = -\pi$$

<u>Assignment</u>

1. Find $\int_c \bar{F}.d\bar{r}$ where $\bar{F}= x^2y^2\bar{i}+y\bar{i}$ and the curve $y^2=4x$ in the xy-plane from (0,0) to (4,4).
2. If $\bar{F}=3xy\bar{i}-5z\bar{j}+10x\bar{k}$ evaluate $\int_C \bar{F}.d\bar{r}$ along the curve $x=t^2+1, y=2t^2, z = t^3$ from t = 1 to t= 2.
3. If $\bar{F}=y\bar{i}+z\bar{j}+x\bar{k}$, find the circulation of $\bar{F}$ round the curve c where c is the circule $x^2 +y^2 =1$, z=0.

4. (i) If $\phi = x^2yz^3, evaluate \int_c \phi d\bar{r}$ along with curve x= t, y =2t, z=3t from t = 0 to t=1.

   (ii) If $\phi = 2xy^2z + x^2y, evaluate \int_c \phi d\bar{r}$ where c is the curve x=t, y=t^2, z= t^3 from t=0 to t=1.

5. (i) Find the work done by the force $\bar{F} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ in taking particle from (1,1,1) to 3,-5,7).

   (ii) Find the work done by the force $\bar{F} = (2y+3)i + (zx)j + (yz - x)k$ when it moves a particle from the point (0,0,0) to (2,1,1) along the curve x = $2t^2$, y = t, z=t^3

2. **Surface integral:** $\int_c \bar{F}.\bar{n}ds$ is called surface integral problems.

Problem 1 : Evaluate $\int \bar{F}.ndS$ where $\bar{F}$ = zi + xj– $3y^2zk$ and S is the surface $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.

Sol. The surface S is $x^2 + y^2 = 16$ included in the first octant between z = 0 and z = 5.

Let $\phi = x^2 + y^2 = 16$

Then $\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial x} + k\frac{\partial\phi}{\partial x} = 2xi + 2yj$

$\therefore$ unit normal $\bar{n} = \frac{\nabla\phi}{|\nabla\varphi|} = \frac{xi + yj}{4}$ $(\because x^2 + y^2 = 16)$

Let R be the projection of S on yz plane

Then $\int_S \bar{F}.\bar{n}dS = \iint_R \bar{F}.\bar{n}\frac{dydz}{|\bar{n}\ .\ \bar{i}|}$ .... *

Given $\bar{F} = zi + xj - 3y^2zk$

$\therefore$ $\bar{F}\ .\ \bar{n} = \frac{1}{4}(xz + xy)$

and $\bar{n}\ .\ \bar{i} = \frac{x}{4}$

In yz plane, x = 0, y = 4

In first octant, y varies from 0 to 4 and z varies from 0 to 5.

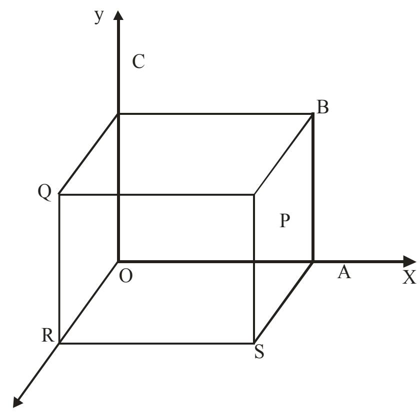
$$\int_S \bar{F}.\bar{n}dS = \int_{y=0}^{4}\int_{z=0}^{5}\left(\frac{xz + xy}{4}\right)\frac{dydz}{\left|\frac{x}{4}\right|}$$

$$= \int_{y=0}^{4}\left(\int_{z=0}^{5}(y + dz)dz\right)dy$$

$$= 90.$$

Problem 2 : If $\bar{F} = zi + xj - 3y^2zk$, evaluate $\int_S \bar{F}.\bar{n}dS$ where S is the surface of the cube bounded by x = 0, x = a, y = 0, y= a, z = 0, z = a.

Sol. Given that S is the surface of the x = 0, x = a, y = 0, y = a, z = 0, z = a, and $\bar{F} = zi + xj - 3y^2zk$ we need to evaluate $\int_S \bar{F}.ndS$.



(i) For OABC

Eqn is z = 0 and dS = dxdy

$$\bar{n} = -\bar{k}$$

$$\int_{S_1} \bar{F}.ndS = -\int_{x=0}^{a} -\int_{y=0}^{a} (yz)\, dxdy = 0$$

(ii) For PQRS

Eqn is z = a and dS = dxdy

$$\bar{n} = \bar{k}$$

$$\int_{S_2} \bar{F}.ndS = \int_{x=0}^{a} \left(\int_{y=0}^{a} y(a)dy \right) dx = \frac{a^4}{2}$$

(iii) For OCQR

Eqn is x = 0, and $\bar{n} = -\bar{i}$, dS = dydz

$$\int_{S_3} \bar{F}.ndS = \int_{y=0}^{a} \int_{z=0}^{a} 4xzdydz = 0$$

(iv) For ABPS

Eqn is x = a, and $\bar{n} = -\bar{i}$, dS = dydz

$$\int_{S_3} \bar{F}.\bar{n}dS = \int_{y=0}^{a} \left(\int_{z=0}^{a} 4azdz \right) dy = 2a^4$$

(v) For OASR

Eqn is $y = 0$, and $\bar{n} = -\bar{j}$, $dS = dxdz$

$$\int_{S_5} \bar{F}.ndS = \int_{y=0}^{a} \int_{z=0}^{a} y^2 dzdx = 0$$

(vi) For PBCQ

Eqn is $y = a$, and $\bar{n} = -\bar{j}$, $dS = dxdz$

$$\int_{S_6} \bar{F}.\bar{n}dS = -\int_{y=0}^{a} \int_{z=0}^{a} y^2 dzdx = 0$$

From (i) – (vi) we get

$$\int_{S_6} \bar{F}.\bar{n}dS = 0 + \frac{a^4}{2} + 0 + 2a^4 + 0 - a4 = \frac{3a^4}{2}$$

3. VOLUME INTEGRALS

Let V be the volume bounded by a surface $\bar{r} = \bar{f}$ (u,v). Let $\bar{F}(\bar{r})$ be a vector point function define over V. Divide V into m sub-regions of volumes $\delta V_1, \delta V_2, \ldots \delta V_p \ldots \delta V_m$

Let P_i ($\bar{r}_i$) be a point in δV_r then form the sum $I_m = \sum_{i=1}^{m} \bar{F}(r1)\delta V_1$. Let m $\rightarrow \infty$ in such a way that δV_1 shrinks to a point,. The limit of I_m if it exists, is called the volume integral of $\bar{F}(\bar{r})$ in the region V is denoted by $\int_V \bar{F}(\bar{r})\,dv$ or $\int_V \bar{F} dv.$

Cartesian form : Let $\bar{F} = (r)i = F_1 \bar{i} + F_2 \bar{i} + F_3 \bar{k}$ where F_1, F_2, F_3 are functions of x,y,z. We know that

dv = dx dy dz. The volume integral given by

$\int_v \bar{F} dv = \iiint F_1 \; (F_1 \bar{i} + F_2 \bar{i} + F_3 \bar{k})$ dx dy dz = $\bar{i} = \iiint F_1$ dxdydz + $\bar{j} = \iiint F_2$ dxdydz + $\bar{k} = \iiint F_3$ dxdydz

Example 2 : If $\overline{F}=(2x^2-3z)\overline{i}-2xy\overline{j}-4x\overline{k}$ then evaluate (*i*) $\int_V \nabla.\overline{F}\,dv$ and (*ii*) $\int_V \nabla\times\overline{F}dv$

; V is the closed region bounded by $x=0$, $y=0$, $z=0$, $2x+2y+z=4$.

Solution : (*i*) $\nabla\cdot\overline{F}=\overline{i}\cdot\dfrac{\partial\overline{F}}{\partial x}+\overline{j}\cdot\dfrac{\partial\overline{F}}{\partial y}+\overline{k}\cdot\dfrac{\partial\overline{F}}{\partial z}=4x-2x=2x.$

The limits are : $z=0$ to $z=4-2x-2y$, $y=0$ to $\dfrac{4-2x}{2}$ (*i.e.*) $2-x$ and $x=0$ to $\dfrac{4}{2}$ (*i.e.*) 2

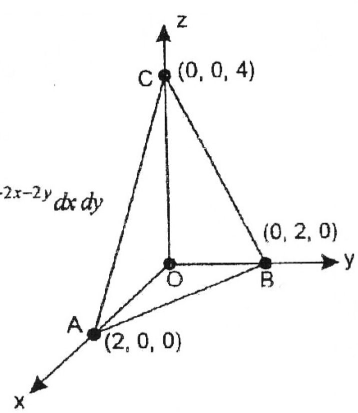
$$\therefore \int_V \nabla.\overline{F}dv=\int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y}2x\,dx\,dy\,dz=\int_{x=0}^{2}\int_{y=0}^{2-x}(2x)(z)_0^{4-2x-2y}dx\,dy$$

$$=\int_{x=0}^{2}\int_{y=0}^{2-x}2x(4-2x-2y)dx\,dy=4\int_{x=0}^{2}\int_{y=0}^{2-x}(2x-x^2-xy)dx\,dy$$

$$=4\int_0^2\left(2xy-x^2y-\frac{xy^2}{2}\right)_0^{2-x}dx=4\int_0^2\left[(2x-x^2)(2-x)-\frac{x}{2}(2-x)^2\right]dx$$

$$=\int_0^2(2x^3-8x^2+8x)dx=\left[\frac{x^4}{2}-\frac{8x^3}{2}+4x^2\right]_0^2=\frac{8}{3}$$

$$(ii)\quad \nabla\times\overline{F}=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2x^2-3z & -2xy & -4x\end{vmatrix}=\overline{j}-2y\overline{k}$$



$$\therefore \int_V \nabla\times\overline{F}\,dv=\iiint_V(\overline{j}-2y\overline{k})dx\,dy\,dz=\int_{x=0}^{2}\int_{y=0}^{2-x}(\overline{j}-2y\overline{k})(z)_0^{4-2x-2y}dx\,dy$$

$$=\int_{x=0}^{2}\int_{y=0}^{2-x}(\overline{j}-2y\overline{k})(4-2x-2y)dx\,dy$$

$$=\int_{x=0}^{2}\int_{y=0}^{2-x}\left\{\overline{j}[(4-2x)-2y]-\overline{k}\left[(4-2x)\cdot 2y-4y^2\right]\right\}dx\,dy$$

$$=\int_{x=0}^{2}\overline{j}\left[(4-2x)y-y^2\right]_0^{2-x}dx-\overline{k}\int_{x=0}^{2}\left[(4-2x)y^2-\frac{4y^3}{3}\right]_0^{2-x}dx$$

$$= \bar{j}\int_0^2 (2-x)^2 dx - \bar{k}\int_0^2 \frac{2}{3}(2-x)^3 dx$$

$$= \bar{j}\left[\frac{(2-x)^3}{-3}\right]_0^2 - \frac{2\bar{k}}{3}\left[\frac{(2-x)^4}{-4}\right]_0^2 = \frac{8}{3}(\bar{j}-\bar{k})$$

EXERCISE 12.3

(1) Evaluate $\iiint (2x+y)dv$ where V is the closed region bounded by the cylinder $z=4-x^2$, and planes $x=0,\ y=0,\ y=2,$ and $z=0$.

(2) If $\phi = 45x^2y$ evaluate $\iiint_V \phi dv$ where V is the closed region bounded by the planes $4x+2y+z=8, y=0, z=0$.

(3) Evaluate $\int_V \bar{F}dv$ when $\bar{F} = x\bar{i} + y\bar{j} + z\bar{k}$ and V is the region bounded by $x=0, y=0, y=6, z=4, z=x^2$.

ANSWERS

(1) $\frac{80}{3}$ (2) 128 (3) $24\bar{i} + 96\bar{j} + \frac{384}{5}\bar{k}$

2. Vector Integral Theorems

Introduction

In this chapter we discuss three important vector integral theorems: (i) Gauss divergence theorem, (ii) Green's theorem in plane and (iii) Stokes theorem. These theorems deal with conversion of

(i) $\int_S \bar{F}.\bar{n}ds$ into a volume integral where S is a closed surface.

(ii) $\int_C \bar{F}.d\bar{r}$ into a double integral over a region in a plane when C is a closed curve in the plane and.

(iii) $\int_S (\nabla x \bar{A}).\bar{n}ds$ into a line integral around the boundary of an open two sided surface.

In solid mechanics, fluid mechanics, quantum mechanics, electrical engineering and various other fields, these theorems will be of great use. Evaluation of an integral of one type may be difficult and using one of the appropriate theorems we may be able to evaluate to the equivalent integral easily. Hence readers are advised to grasp the significance in each case.

I. GAUSS'S DIVERGENCE THEOREM

(Transformation between surface integral and volume integral)

Let S be a closed surface enclosing a volume v. if $\bar{F}$ is a continuously differentiable vector point function, then

$$\int_V div\,\bar{F}\,dv = \int_S \bar{F}.\bar{n}\; dS$$

When $\bar{n}$ is the outward drawn normal vector at any point of S.

SOLVED EXAMPLES

Example 1: Verify Gauss Divergence theorem for $\bar{F} = (x^3 - yz)\bar{i} - 2x^2 y\bar{j} + z\bar{k}$ taken over the surface of the cube bounded by the planes x = y = z = a and coordinate planes.

Sol: By Gauss Divergence theorem we have

$$\int \bar{F}.\bar{n}dS = \int_v div\ \bar{F}\ dv$$

$$RHS = \int_0^a\int_0^a\int_0^a (3x^2-2x^2+1)dx\ dy\ dz = \int_0^a\int_0^a\int_0^a (x^2+1)\,dx\ dy\ dz = \int_0^a\int_0^a \left(\frac{x^3}{3}+x\right)_0^a dy\ dz$$

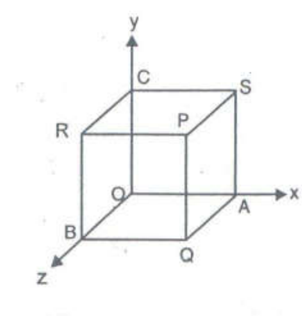
$$= \int_0^a\int_0^a \left[\frac{a^3}{3}+a\right]dy\ dz = \int_0^a \left[\frac{a^3}{3}+a\right](y)_0^a\ dz = \left(\frac{a^3}{3}+a\right)a\int_0^a dz = \left(\frac{a^3}{3}+a\right)(a^2)$$

$$= \frac{a^3}{3}+a^3 \ldots(1)$$

Verification: We will calculate the value of $\int_S F.\bar{n}dS$ *over the six faces of the cube.*

(i) For S_1 = PQAS; unit outward drawn normal $n = i$

x=a; ds=dy dz; 0≤y≤a, 0≤z≤a



$$\bar{F}.n = x^3 - yz = a^3 - yz\ since\ x = a$$

$$\int_{S_1}\int \bar{F}.\bar{n}dS = \int_{z=0}^a\int_{y=0}^a (a^3-yz)dy\ dz$$

$$= \int_{z=0}^a \left[a^3y - \frac{y^2}{2}z\right]_{y=0}^a dz$$

$$= \int_{z=0}^a \left(a^4 - \frac{a^2}{2}z\right)dz$$

$$= a^5 - \frac{a^4}{4} \ldots(2)$$

(ii) For S_2 = OCRB; unit outward drawn normal $\bar{n} = -i$

x=a; ds=dy dz; 0≤y≤a, y≤z≤a

$$\bar{F}.\bar{n} = -(x^3 - yz) = yz \text{ since } x = 0$$

$$\int_{S_2}\int \bar{F}.\bar{n}dS = \int_{z=0}^{a}\int_{y=0}^{a} yz\,dy\,dz = \int_{z=0}^{a}\left[\frac{y^2}{2}\right]_{y=0}^{a} zdz$$

$$= \frac{a^2}{2}\int_{z=0}^{a} zdz = \frac{a^4}{4} \dots (3)$$

(iii) For S_3 = RBQP; Z = a; ds = dxdy; $\bar{n} = \bar{k}$

$0 \le x \le a,\ 0 \le y \le a$

$$\bar{F}.\bar{n} = z = a \text{ since } z = a$$

$$\int_{S_3}\int \bar{F}.\bar{n}dS = \int_{y=0}^{a}\int_{x=0}^{a} a\,dx\,dy = a^3 \dots (4)$$

(iv) For S_4 = OASC; z = 0; $\bar{n} = -\bar{k}$, ds = dxdy;

$0 \le x \le a,\ 0 \le y \le a$

$$\bar{F}.\bar{n} = -z = 0 \text{ since } z = 0$$

$$\int_{S_4}\int \bar{F}.\bar{n}dS = 0 \dots (5)$$

(v) For S_5 = PSCR; y = a; $\bar{n} = \bar{j}$, ds = dzdx;

$0 \le x \le a,\ 0 \le z \le a$

$$\bar{F}.\bar{n} = -2x^2y = -2ax^2 \text{ since } y = a$$

$$\int_{S_5}\int \bar{F}.\bar{n}dS = \int_{x=0}^{a}\int_{z=0}^{a} (-2ax^2)\,dzdx$$

$$= \int_{x=0}^{a} (-2ax^2)_{z=0}^{a} \, dx$$

$$= -2a^2\left(\frac{x^3}{3}\right)_0^a = \frac{-2a^5}{3} \ldots (6)$$

(vi) For S_6 = OBQA; y = 0; $\bar{n} = -\bar{j}$, ds = dzdx;

$0 \leq x \leq a$, $0 \leq y \leq a$

$$\bar{F}.\bar{n} = 2x^2y = 0 \; since \; y = 0$$

$$\int_{S_6}\int \bar{F}.\bar{n}dS = 0$$

$$\int_{S}\int \bar{F}.\bar{n}dS = \int_{S_1}\int + \int_{S_2}\int + \int_{S_3}\int + \int_{S_4}\int + \int_{S_5}\int + \int_{S_6}\int$$

$$= a^5 - \frac{a^4}{4} - \frac{a^4}{4} + a^3 + 0 - \frac{2a^5}{3} + 0$$

$$= \frac{a^5}{3} + a^3 = \int\int_V\int \bar{\nabla}.\bar{F}\, dv \; using \; (1)$$

Hence Gauss Divergence theorem is verified

Example 2: Compute $\int (ax^2 + by^2 + cz^2) dS$ over the surface of the sphere $x^2+y^2+z^2 = 1$

Sol: By divergence theorem $\int \bar{F}.\bar{n}dS - \int_V \bar{\nabla}.\bar{F}\; dv$

$$Given \; \bar{F}.\bar{n} = ax^2 + by^2 + cz^2. Let \; \phi = x^2 + y^2 + z^2 - 1$$

Normal vector $\bar{n}$ to the surface ϕ is

$$\bar{\nabla}\phi = \left(\bar{i}\frac{\delta}{\delta x} + \bar{j}\frac{\delta}{\delta y} + \bar{k}\frac{\delta}{\delta y}\right)(x^2 + y^2 + z^2 - 1) = 2(x\bar{i} + y\bar{j} + z\bar{k})$$

$$\textit{Unit normal vector} = \bar{n} = \frac{2(x\bar{\imath} + y\bar{\jmath} + z\bar{k})}{2\sqrt{x^2 + y^2 + z^2}} = x\bar{\imath} + y\bar{\jmath} + z\bar{k} \text{ since } x^2 + y^2 + z^2 = 1$$

$$\bar{F}.\bar{n} = \bar{F}.(x\bar{\imath} + y\bar{\jmath} + z\bar{k}) = (ax^2 + by^2 + cz^2) = (ax\bar{\imath} + by\bar{\jmath} + cz\bar{k}).(x\bar{\imath} + y\bar{\jmath} + z\bar{k})$$

$$i.e.\ ax\bar{\imath} + by\bar{\jmath} + cz\bar{k} \quad \bar{\nabla}.\bar{F} = a + b + c$$

Hence by Gauss Divergence theorem,

$$\int_S (ax^2 + by^2 + cz^2)dS = \int_V (a + b + c)dv = (a + b + c)V = \frac{4\pi}{3}(a + b + c)$$

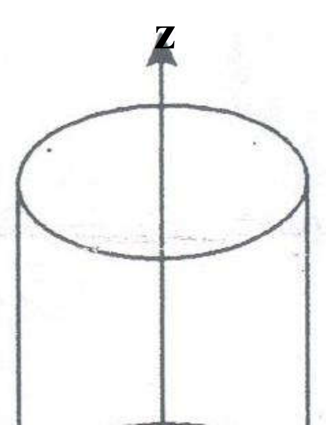
$$\left[\textit{Since } V = \frac{4\pi}{3} \textit{ is the volume of the sphere of unit radius}\right]$$

Example 3: By transforming into triple integral, evaluate $\int\int x^3\, dy\, dz + x^2y\, dz\, dx + x^2 dx\, dy$ where S is the closed surface consisting of the cylinder $x^2+y^2 = a^2$ and the circular discs z= 0 , z= b.

Sol: Here $F_1 = x^3, F_2 = x^2y, F_3 = x^2z \textit{ and } \bar{F} = F_1\bar{\imath} + F_2\bar{\jmath} + F_3\bar{k}$

$$\frac{\delta F_1}{\delta x} = 3x^2, \frac{\delta F_2}{\delta y} = x^2, \frac{\delta F_3}{\delta z} = x^2$$

$$\bar{\nabla}.\bar{F} = \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z} = 3x^2 + x^2 + x^2 = 5x^2$$



x

By Gauss Divergence theorem,

$$\int\int F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy = \int\int\int \left(\frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}\right) dx\, dy\, dz$$

$$\int\int_S (x^3\, dy\, dz + x^2 y\, dz\, dx + x^2 dx\, dy) = \int\int\int 5x^2\, dx\, dy\, dz$$

$$= 5 \int_{-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^{b} x^2 dx\, dy\, dz$$

$$= 20 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2}} x^2\, dx\, dy\, dz\ [Integrand\ is\ even\ function]$$

$$= 20 \int_0^a \int_0^{\sqrt{a^2-x^2}} x^2 (z)_0^b\, dxdy = 20b \int_{x=0}^{a} \int_0^{\sqrt{a^2-x^2}} x^2 dx\, dy$$

$$= 20b \int_{x=0}^{a} x^2 (y)_0^{\sqrt{a^2-x^2}}\, dx = 20b \int_0^a x^2 \sqrt{a^2 - x^2}\, dx$$

$$= 20b \int_0^{\frac{\pi}{2}} a^2 \sin^2\theta \sqrt{a^2 - a^2 \sin^2\theta}\, (a\cos\theta d\theta)$$

$$\left[put\ x = a\sin\theta ==> dx = a\cos\theta\ d\theta\ when\ x = a ==> \theta = \frac{\pi}{2}\ and\ x = 0 ==> \theta = 0\right]$$

$$= 20a^4 b \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta d\theta = 5a^4 b \int_0^{\frac{\pi}{2}} (2\sin\theta\cos\theta)^2\, d\theta = 5a^4 b \int_0^{\frac{\pi}{2}} \frac{1-\cos 4\theta}{2} d\theta$$

$$= \frac{5a^4 b}{2}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{5a^4 b}{2}\left[\frac{\pi}{2}\right] = \frac{5}{4}\pi a^4 b$$

Example 4: Applying Gauss divergence theorem, Prove that

$\int \bar{r} \cdot \bar{n} dS = 3V$ or $\int \bar{r} \cdot d\bar{s} = 3V$

Sol: Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ we know that div $\bar{r} = 3$

By Gauss divergence theorem, $\int F \cdot \bar{n} dS = \int_v div\, F\, dv$

$$Take\ \bar{F} = \bar{r} \Rightarrow \int_S \bar{r} \cdot \bar{n} dS = \int_V 3\, dV = 3V. Hence\ the\ result$$

Example 5: Show that $\int_S (ax\bar{i} + by\bar{j} + cz\bar{k}) \cdot \bar{n} dS = \frac{4\pi}{3}(a+b+c)$, **where S is the surface of the sphere $x^2+y^2+z^2=1$.**

Sol: Take $F = ax\bar{i} + by\bar{j} + cz\bar{k}$

$$div\ \bar{F} = \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z} = a + b + c$$

By Gauss divergence theorem,

$$\int_S \bar{F} \cdot \bar{n} dS = \int_V \bar{V} \cdot \bar{F} dV = (a+b+c) \int_V dV = (a+b+c)V$$

$$We\ have\ V = \frac{4}{3}\pi r^3\ for\ the\ sphere. Here\ r = 1$$

$$\int_S \bar{F} \cdot \bar{n} dS = (a+b+c)\frac{4\pi}{3}$$

Example6: Using Divergence theorem, evaluate

$\int\int_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$, *where S*: **$x^2+y^2+z^2=a^2$**

Sol: we have by Gauss divergence theorem, $\int \bar{F}.\bar{n}dS = \int_v div\, \bar{F}\, dv$

L.H.S can be written as $\int (F_1 dydz + F_2 dzdx + F_3 dxdy)$ in Cartesian form

Comparing with the given expression we have F_1=x, F_2=y, F_3=z

Then $div\, \bar{F} = \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z} = 3$

$$\int_V div\, \bar{F} dv = \int_V 3dv = 3V$$

Here V is the volume of the sphere with radius a.

$$V = \frac{4}{3}\pi a^3$$

Hence $\int\int (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy) = 4\pi a^3$

Example 7: Apply divergence theorem to evaluate

$\int\int_S (x+z)dy\, dz + (y + z_{dz} dx + (x+y)dxdy\ where$ **S is the surface of the sphere $x^2+y^2+z^2=4$**

Sol: Given $\int\int_S (x+z)dy\, dz + (y + z_{dz} dx + (x+y)dxdy$

Here F_1 = x+z, F_2 = y+z, F_3= x+y

$$\frac{\delta F_1}{\delta x} = 1, \frac{\delta F_2}{\delta y} = 1, \frac{\delta F_3}{\delta z} = 0\ and\ \frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z} = 1 + 1 + 0 = 2$$

By Gauss Divrgence theorem,

$$\int\int_S F_1 dydz + F_2 dzdx + F_3 dxdy = \int\int\int_V \left(\frac{\delta F_1}{\delta x} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}\right) dxdydz$$

$$= \int\int\int 2dxdydz = 2\int_V dv = 2V$$

$$= 2\left[\frac{4}{3}\pi(2)^3\right] = \frac{64\pi}{3} \; [for\ the\ sphere, radius = 2]$$

Example 8: Evaluate $\int_S \bar{F}.\bar{n}ds, if\ F = xy\bar{i} + z^2\bar{j} + 2yz\bar{k}$ over the tetrahedron bounded by x=0, y=0, z=0 and the plane x+y+z=1.

Sol: Given F = $xy\bar{i} + z^2\bar{j} + 2yz\bar{k}$, then div. F = y+2y = 3y

$$\int \bar{F}.\bar{n}dS = \int_v div\ \bar{F}\ dv = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} 3y\ dx\ dy\ dz$$

$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x} y[z]_0^{1-x-y} dxdy = 3\int_{x=0}^{1}\int_{y=0}^{1-x} y(1-x-y)dx\ dy$$

$$= 3\int_{x=0}^{1}\left[\frac{y^2}{2} - \frac{xy^2}{2} - \frac{y^3}{3}\right]_0^{1-x} dx = 3\int_0^1 \left[\frac{(1-x)^2}{2} - \frac{x(1-x)^2}{2} - \frac{(1-x)^3}{3}\right] dx$$

$$= 3\int_0^1 \left[\frac{(1-x)^3}{2} - \frac{(1-x)^3}{3}\right] dx = 3\int_0^1 \frac{(1-x)^3}{6} dx = \frac{3}{6}\left[\frac{-(1-x)^4}{4}\right]_0^1 = \frac{1}{8}$$

Example 9: Use divergence theorem to evaluate $\int_S \int F.dS\ where\ F =$ x^3i+y^3j+z^3k and S is the surface of the sphere $x^2+y^2+z^2 = r^2$

Sol: We have

$$\bar{\nabla}.\bar{F} = \frac{\delta}{\delta x}(x^3) + \frac{\delta}{\delta y}(y^3) + \frac{\delta}{\delta z}(z^3) = 3(x^2 + y^2 + z^2)$$

By divergence theorem,

$\bar{\nabla}.\bar{F}\,dV = \int\int_V \int \bar{\nabla}.\bar{F}\,dV = \int\int_V \int 3(x^2+y^2+z^2)dx\,dy\,dz$

$$= 3\int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} r^2(r^2\sin\theta\,dr\,d\theta\,d\phi)$$

[Changing into spherical polar coordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$]

$$\int_S\int \bar{F}.dS = 3\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4\sin\theta\left[\int_{\phi=0}^{2\pi} d\phi\right]dr\,d\theta$$

$$3\int_{r=0}^{a}\int_{\theta=0}^{\pi} r^4\sin\theta\,(2\pi-0)\,dr\,d\theta = 6\pi\int_{r=0}^{a} r^4\left[\int_0^{\pi}\sin\theta\,d\theta\right]dr$$

$$= 6\pi\int_{r=0}^{a} r^4(-\cos\theta)_0^{\pi}\,dr = -6\pi\int_0^{a} r^4(\cos\pi-\cos 0)\,dr$$

$$= 12\pi\int_0^{a} r^4 dr = 12\pi\left[\frac{r^5}{5}\right]_0^a = \frac{12\pi a^5}{5}$$

Example 10: Use divergence theorem to evaluate $\int\int_S \bar{F}.dS$ where $\bar{F} = 4xi - 2y^2j + z^2k$ and S is the surface bounded by the region $x^2+y^2=4$, $z=0$ and $z=3$.

Sol: We have

$$div\,\bar{F} = \bar{\nabla}.F = \frac{\delta}{\delta x}(4x) + \frac{\delta}{\delta y}(-2y^2) + \frac{\delta}{\delta z}(z^2) = 4 - 4y + 2z$$

By divergence theorm,

$$\int_S\int \bar{F}.dS = \int\int_V\int \bar{\nabla}.\bar{F}\,dV$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{3} (4 - 4y + 2z) dx\, dy\, dz$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [(4 - 4y)z + z^2]_0^3\, dx\, dy$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [12(1 - y) + 9]\, dx\, dy$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y) dx\, dy$$

$$= \int_{-2}^{2} \left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 21\, dy - 12 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} y dy \right] dx$$

$$= \int_{-2}^{2} \left[21X2 \int_{0}^{\sqrt{4-x^2}} dy - 12(0) \right] dx$$

[*since tge integrans in forst integral is even and in 2nd integral it is on odd function*]

$$= 42 \int_{-2}^{2} (y)_0^{\sqrt{4-x^2}}\, dx$$

$$= 42 \int_{-2}^{2} \sqrt{4 - x^2} dx = 42X2 \int_{0}^{2} \sqrt{4 - x^2}\, dx$$

$$= 84 \left[\frac{x}{2} \sqrt{4 - x^2} + \frac{4}{2} \sin^{-1} \frac{x}{2} \right]_0^2$$

$$= 84 \left[0 + 2.\frac{\pi}{2} - 0 \right] = 84\pi$$

Example 11: Verify divergence theorem for $\bar{F} = x^2 i + y^2 j + z^2 k$ over the surface S of the solid cut off by the plane x+y+z=a in the first octant.

Sol; By Gauss theorem, $\int \bar{F}.\bar{n}dS = \int_v \; div\,\bar{F}\; dv$

$Let\ \phi = x + y + z - a\ be\ the\ given\ plane\ then$

$$\frac{\delta\phi}{\delta x} = 1, \frac{\delta\phi}{\delta y} = 1, \frac{\delta\phi}{\delta z} = 1$$

$$grad\ \phi = \sum \bar{\imath}\frac{\delta\phi}{\delta x} = \bar{\imath} + \bar{\jmath} + \bar{k}$$

$$Unit\ normal = \frac{grad\ \phi}{|grad\ \phi|} = \frac{\bar{\imath} + \bar{\jmath} + \bar{k}}{\sqrt{3}}$$

Let R be the projection of S on xy-plane

Then the equation of the given plane will be x+y=a ➔ y=a-x

Also when y=0, x=a

$$\int \bar{F}.\bar{n}dS = \int_R \int \frac{\int \bar{F}.\bar{n}\ dx\ dy}{|\bar{n}.\bar{k}|}$$

$$= \int_{x=0}^{a}\int_{y=0}^{a-x} \frac{\frac{x^2+y^2+z^2}{\sqrt{3}}}{1/\sqrt{3}} dx\ dy \quad = \int_0^a \int_{y=0}^{a-x} [x^2 + y^2 + (a-x-y)^2] dx\ dy\ [since\ x+y+z=a]$$

$$= \int_0^a \int_0^{a-x} [2x^2 + 2y^2 - 2ax + 2xy - 2ay + a^2]\, dx\ dy$$

$$= \int_{x=0}^{a} \left[2x^2 y + \frac{2y^3}{3} + xy^2 - 2axy - ay^2 + a^2 y\right]_0^{a-x} dx$$

$$= \int_{x=0}^{a} [2x^2(a-x) + \frac{2}{3}(a-x)^3 + x(a-x)^2 - 2ax(a-x) - a(a-x)^2 + a^2(a-x) dx$$

$$\int \bar{F}.\bar{n}dS = \int_0^a \left(-\frac{5}{3}x^3 + 3ax^2 - 2a^2 x + \frac{2}{3}a^3\right) dx = \frac{a^4}{4}, on\ simplification\ ...(1)$$

Given $\bar{F} = x^2 i + y^2 j + z^2 k$

$$div\ \bar{F} = \frac{\delta}{\delta x}(x^2) + \frac{\delta}{\delta y}(y^2) + \frac{\delta}{\delta z}(z^2) = 2(x+y+z)$$

$$Now \iiint div\ \bar{F}.dv = 2 \int_{x=0}^{a} \int_{y=0}^{a-x} \int_{z=0}^{a-x-a} (x+y+z)\,dx\,dy\,dz$$

$$= 2 \int_{x=0}^{a} \int_{y=0}^{a-x} \left[z(x+y) + \frac{z^2}{2}\right]_0^{a-x-y} dx\,dy$$

$$= 2 \int_{x=0}^{a} \int_{y=0}^{a-x} (a-x-y)\left[x+y+\frac{a-x-y}{2}\right] dx\,dy$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} (a-x-y)[a+x+y]\,dx\,dy$$

$$= \int_0^a \int_0^{a-x} [a^2-(x+y)^2]\ dy\,dx = \int_0^a \int_0^{a-x} (a^2-x^2-y^2-2xy)\,dx\,dy$$

$$= \int_0^a \left[a^2y - x^2y - \frac{y^3}{3} - xy^2\right]_0^{a-x} dx$$

$$= \int_0^a (a-x)(2a^2-x^2-ax)\,dx = \frac{a^4}{4} \ldots\ldots(2)$$

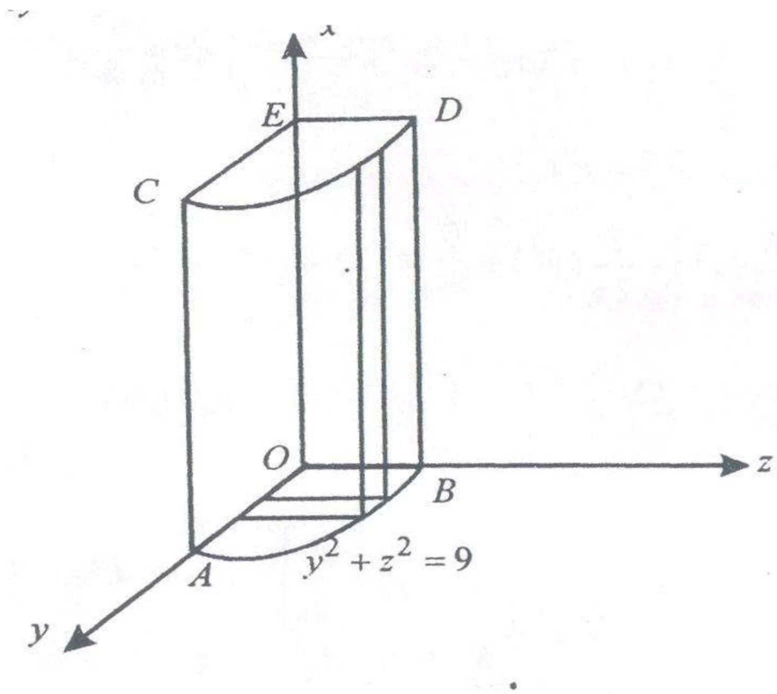
Hence from (1) and (2), the Gauss Divergence theorem is verified.

Example 12: Verify divergence theorem for $2x^2yi-y^2j+4xz^2k$ ove the region of first octant of the cylinder $y^2+z^2=9$ and $x=2$.

(or) Evaluate $\int_S \int \bar{F}.\bar{n}ds,\ where\ \bar{F} - 2x^2yi - y^2j + 4xz^2k$ and S is the closed surface of the region in the octant bounded by the cylinder $y^2+z^2 = 9$ and the planes x=0, x=2, y=0, z=0

Sol: Let $\bar{F} = 2x^2yi - y^2j + 4xz^2k$

$$\bar{\nabla}.\bar{F} = \frac{\delta}{\delta x}(2x^2) + \frac{\delta}{\delta y}(-y^2) + \frac{\delta}{\delta z}(4xz^2) = 4xy - 2y + 8xz$$



$$\iiint_V \bar{\nabla}.\bar{F}\,dv = \int_{x=0}^{2}\int_{y=0}^{3}\int_{z=0}^{\sqrt{9-y^2}} (4xy-2y+8xz)dz\,dy\,dx$$

$$= \int_0^2\int_0^3 \left[(4xy-2y)z+8x\,\frac{z^2}{2}\right]_{z=0}^{\sqrt{9-y^2}} dy\,dx$$

$$= \int_0^2\int_0^3 \left[(4xy-2y)\sqrt{9-y^2}+4x(9-y^2)\right]dy\,dx$$

$$= \int_0^2\int_0^3 [(1-2x)(-2y)\sqrt{9-y^2}+4x(9-y^2)]\,dy\,dx$$

$$= \int_0^2\left\{\left[(1-2x)\frac{(9-y^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^3+4x\left(9y-\frac{y^3}{3}\right)_0^3\right\}dx$$

$$= \int_0^2\{\frac{2}{3}(1-2x)[0-27]+4x[27-9]\}\,dx = \int_0^2[-18(1-2x)+72x]dx$$

$$= [-18(x-x^2)+72\frac{x^2}{2}]_0^2 = -18(2-4)+36(4) = 36+144 = 180\,\dots\dots(1)$$

$Now\ we\ sall\ calculate \int_S \bar{F}.\bar{n}\ ds\ \ for\ all\ the\ five\ faces.$

$$\int_S \bar{F}.\bar{n}\,ds = \int_{S_1} \bar{F}.\bar{n}\,ds + \int_{S_2} \bar{F}.\bar{n}\,ds + \cdots + \int_{S_5} \bar{F}.\bar{n}\,ds$$

Where S_1 is the face OAB, S_2 is the face CED, S_3 is the face OBDE, S_4 is the face OACE and S_5 is the curved surface ABDC.

$(i) On\ S_1 : x = 0, \bar{n} = -i. \qquad \bar{F}.\bar{n} = 0 \qquad Hence \int_{S_1} \bar{F}.\bar{n}ds = 0$

$(ii)\ On\ S_2 : x = 2, \bar{n} = i. \qquad \bar{F}.\bar{n} = 8y$

$$\int_{S_2} \bar{F}.\bar{n}ds = \int_0^3 \int_0^{\sqrt{9-z^2}} 8y\,dy\,dz = \int_0^3 8\left(\frac{y^2}{2}\right)_0^{\sqrt{9-z^2}} dz$$

$$= 4\int_0^3 (9 - z^2)dz = 4\left(9z - \frac{z^3}{3}\right)_0^3 = 4(27 - 9) = 72$$

$(iii) On\ S_3 : y = 0, \bar{n} = -j. \qquad \bar{F}.\bar{n} = 0. \qquad Hence \int_{S_3} \bar{F}.\bar{n}ds = 0$

$(iv) On\ S_4 : z = 0, \bar{n} = -k. \qquad \bar{F}.\bar{n} = 0. \qquad Hence \int_{S_4} \bar{F}.\bar{n}ds = 0$

$$(v) On\ S_5 : y^2 + z^2 = 9, \bar{n} = \frac{\bar{\nabla}\ (y^2 + z^2)}{|\bar{\nabla}\ (y^2 + z^2)|} = \frac{2yj + 2zk}{\sqrt{4y^2 + 4z^2}} = \frac{yj + zk}{\sqrt{4X9}} = \frac{yj + zk}{3}$$

$$\bar{F}.\bar{n} = \frac{-y^3 + 4xz^3}{3}\ and\ \bar{n}.k = \frac{z}{3} = \frac{1}{3}\sqrt{9 - y^2}$$

Hence $\int_{S_5} \bar{F}.\bar{n}ds = \int\int_R \bar{F}.\bar{n}\ \frac{dx\,dy}{|\bar{n}.\bar{k}|}$ $Where\ R\ is\ the\ projection\ of\ S_5\ on\ xy - plane.$

$$= \int_R \int \frac{4xz^3 - y^3}{\sqrt{9-y^2}} dx\, dy = \int_{x=0}^{2} \int_{y=0}^{3} \left[4x(9-y^2) - y^3 \left(9 - y^2\right)^{-\frac{1}{2}}\right] dy\, dx$$

$$= \int_0^2 72x\, dx - 18 \int_0^2 dx = 72\left(\frac{x^2}{2}\right)_0^2 - 18(x)_0^2 = 144 - 36 = 108$$

$Thus \int_S \bar{F}.\bar{n} ds = 0 + 72 + 0 + 0 + 108 = 180$ … … (2)

Hence the Divergence theorem is verified from the equality of (1) and (2).

Example 13: Use Divergence theorem to evaluate $\int\int (x\bar{\imath} + y\bar{\jmath} + z^2\bar{k}).\bar{n}.ds$ Where S is the surface bounded by the cone $x^2+y^2=z^2$ in the plane z = 4.

Sol: Given $\int\int (x\bar{\imath} + y\bar{\jmath} + z^2\bar{k}).\bar{n}.ds$ Where S is the surface bounded by the cone $x^2+y^2=z^2$ in the plane z = 4.

Let $\bar{F} = x\bar{\imath} + y\bar{\jmath} + z^2\bar{k}$

By Gauss Divergence theorem, we have

$$\int\int (x\bar{\imath} + y\bar{\jmath} + z^2\bar{k}).\bar{n}.ds = \int\int_V \int \bar{\nabla}.\bar{F}\, dv$$

Now $\bar{\nabla}.\bar{F} = \frac{\delta}{\delta x}(x) + \frac{\delta}{\delta y}(y) + \frac{\delta}{\delta z}(z^2) = 1 + 1 + 2z = 2(1+z)$

On the cone $x^2 + y^2 = z^2$ *and* $z = 4 ==> x^2 + y^2 = 16$

The limits are $z = 0$ *to* $4, y = o$ *to* $\sqrt{16-x^2}, x = 0$ *to* $4.$

$$\int\int_V \int \bar{\nabla}.\bar{F}\, dv = \int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^4 2(1+z) dx\, dy\, dz$$

$$= 2 \int_0^4 \int_0^{\sqrt{16-x^2}} \left\{[z]_0^4 + \left[\frac{z^2}{2}\right]_0^4\right\} dx\, dy$$

$$= 2\int_0^4\int_0^{\sqrt{16-x^2}} [4+8]dx\,dy = 2X12\int_0^4 [y]_0^{\sqrt{16-x^2}}\,dx$$

$$= 24\int_0^4 \sqrt{16-x^2}\,dx = 24\int_0^{\frac{\pi}{2}} \sqrt{16-16\,sin^2\,\theta}\;.4\cos\theta d\theta$$

$\left[put\ x = 4\sin\theta ==> dx = 4\cos\theta\ \ d\theta. Also\ z = 0 ==>\ \theta = 0\ and\ x = 4 ==>\ \theta = \frac{\pi}{2}\right]$

$$\int\int_V\int \bar{V}.\bar{F}\,dv = 96\,X\,4\int_0^{\frac{\pi}{2}} 4\sqrt{1-sin\ ^2\theta}\ \ cos\theta d\theta = 96\,X\,4\int_0^{\frac{\pi}{2}} cos^2\theta\ \ d\theta$$

$$= 96\,X\,4\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}d\theta = 96X4\int_0^{\frac{\pi}{2}}\left[\frac{1}{2}+\frac{\cos 2\theta}{2}\right]d\theta$$

$$= 384\left[\frac{1}{2}\theta+\frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = 96\pi$$

Example 14: Use Gauss Divergence theorem to evaluate $\int\int_S (yz^2\bar{\imath}+zx^2\bar{\jmath}+2z^2\bar{k}).ds, where$ S is the closed surface bounded by the xy plane and the upper half of te sphere $x^2+y^2+z^2=a^2$

Sol: Divergence theorem states that

$$\int\int_S \bar{F}.ds = \int\int_V\int \bar{V}.\bar{F}\,dv$$

$$Here\ \bar{V}.\bar{F} = \frac{\delta}{\delta x}(yz^2)+\frac{\delta}{\delta y}(zx^2)+\frac{\delta}{\delta z}(2z^2) = 4z$$

$$\int\int_S \bar{F}.ds = \int\int_V\int 4z\,dx\,dy\,dz$$

$Introducing\ spheical\ polar\ coordinates\ x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta\ \ then\ dx\,dy\,dz = \ r^2\ \ dr\ d\theta\,d\phi$

$$\int\int_S \bar{F}.ds = 4 \int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} (r\cos\theta)\ (r^2 \sin\theta\ dr\ d\theta\ d\phi)$$

$$= 4 \int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3 \sin\theta\cos\theta \left[\int_{\phi=0}^{2\pi} d\phi\right] dr\ d\theta$$

$$= 4. \int_{r=0}^{a}\int_{\theta=0}^{\pi} r^3 \sin\theta\cos\theta\ (2\pi - 0) dr\ d\theta$$

$$= 4\pi \int_{r=0}^{a} r^3 \left[\int_0^{\pi} \sin 2\theta\ d\theta\right] dr = 4\pi \int_{r=0}^{a} r^3 \left(-\frac{\cos 2\theta}{2}\right)_0^{\pi} dr$$

$$= (-2\pi)\int_0^a r^3 (1-1) dr = 0$$

Example 15: Verify Gauss divergence theorem for $\bar{F} = x^3\bar{\imath} + y^3\bar{\jmath} + z^3\bar{k}$ taken over the cuve bounded by x = 0, x = a, y= 0, y = a, z = 0, z = a.

Sol: We have $F = x^3\bar{\imath} + y^3\bar{\jmath} + z^3\bar{k}$

$$\bar{\nabla}.\bar{F} = \frac{\delta}{\delta x}(x^3) + \frac{\delta}{\delta y}(y^3) + \frac{\delta}{\delta z}(z^3) = 3x^2 + 3y^2 + 3z^2$$

$$\int\int_V\int \bar{\nabla}.\bar{F}\, dv = \int\int_V\int (3x^2 + 3y^2 + 3z^2) dx\ dy\ dz$$

$$= 3 \int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a} (x^2 + y^2 + z^2) dx\ dy\ dz$$

$$= 3 \int_{z=0}^{a}\int_{y=0}^{a} \left(\frac{x^3}{3} + xy^2 + z^2 x\right)_0^a dy\ dz$$

$$= 3 \int_{z=0}^{a} \int_{y=0}^{a} \left(\frac{a^3}{a} + ay^2 + az^2 \right) dy\, dz$$

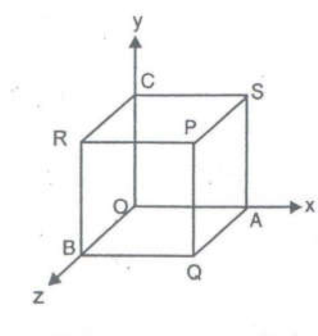
$$= 3 \int_{z=0}^{a} \left(\frac{a^3}{3} y + a\frac{y^3}{3} + az^2 y \right)_0^a dz$$

$$= 3 \int_0^a \left(\frac{a^4}{3} + \frac{a^4}{3} + a^2 z^2 \right) dz = 3 \int_0^a \left(\frac{2}{3} a^4 + a^2 z^2 \right) dz$$

$$= 3 \left(\frac{2}{3} a^4 z + a^2 . \frac{z^3}{3} \right)_0^a = 3 \left(\frac{2}{3} a^5 + \frac{1}{3} a^5 \right)$$

$$= 3a^5$$

To evaluate the surface integral divide the closed surface S of the cube into 6 parts.



i.e., S_1 : The face DEFA ; S_4 : The face OBDC

S_2 : The face AGCO ; S_5 : The face GCDE

S_3 : The face AGEF ; S_6: The face AFBO

$$\int_S \int \bar{F}.\bar{n} ds = \int_{S_1} \int \bar{F}.\bar{n} ds + \int_{S_2} \int \bar{F}.\bar{n} ds + \cdots + \int_{S_6} \int \bar{F}.\bar{n} ds$$

$On\ S_1, we\ have\ \bar{n} = \bar{\imath}, x = a$

$$\int_{S_1} \int \bar{F}.\bar{n} ds = \int_{z=0}^{a} \int_{y=0}^{a} \left(a^3 \bar{\imath} + y^3 \bar{\jmath} + z^3 \bar{k} \right) . \bar{\imath}\, dy\, dz$$

$$= \int_{z=0}^{a}\int_{y=0}^{a} a^3 dy\, dz = a^3 \int_{0}^{a} (y)_0^a \, dz$$

$$= a^4 (z)_0^a = a^5$$

$On\ S_2, we\ have\ \bar{n} = -\bar{\imath}, x = 0$

$$\iint_{S_2} \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{y=0}^{a} (y^3\bar{\jmath} + z^3\bar{k}).(-\bar{\imath})dy\, dz = 0$$

$On\ S_3, we\ have\ \bar{n} = \bar{\jmath}, y = a$

$$\iint_{S_3} \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{x=0}^{a} (x^3\bar{\imath} + a^3\bar{\jmath} + z^3\bar{k}).\bar{\jmath}\, dx\, dz = a^3 \int_{z=0}^{a}\int_{x=0}^{a} dx\, dz = a^3 \int_{0}^{a} a\, dz = a^4 (z)_0^a$$
$$= a^5$$

$On\ S_4, we\ have\ \bar{n} = -\bar{\jmath}, y = 0$

$$\iint_{S_4} \bar{F}.\bar{n}ds = \int_{z=0}^{a}\int_{x=0}^{a} (x^3\bar{\imath} + z^3\bar{k}).(-\bar{\jmath})dx\, dz = 0$$

$On\ S_5, we\ have\ \bar{n} = \bar{k}, z = a$

$$\iint_{S_5} \bar{F}.\bar{n}ds = \int_{y=0}^{a}\int_{x=0}^{a} (x^3\bar{\imath} + y^3\bar{\jmath} + a^3\bar{k}).\bar{k}\, dx\, dy$$

$$= \int_{y=0}^{a}\int_{x=0}^{a} a^3 dx\, dy = a^3 \int_{0}^{a} (x)_0^a dy = a^4 (y)_0^a = a^5$$

$On\ S_6, we\ have\ \bar{n} = -\bar{k}, z = 0$

$$\iint_{S_6} \bar{F}.\bar{n}ds = \int_{y=0}^{a}\int_{x=0}^{a} (x^3\bar{\imath} + y^3\bar{\jmath}).(-\bar{k})dx\, dy = 0$$

$$Thus \int_S \int \bar{F}.\bar{n}ds = a^5 + 0 + a^5 + 0 + a^5 + 0 = 3a^5$$

$$Hence \int_S \int \bar{F}.\bar{n}ds = \int_V \int \bar{\nabla}.\bar{F}\, dv$$

The Gauss divergence theorem is verified.

Assignment

1. Evaluate $\iint_s xdy\, dz + y\, dz\, dx + zdxdy$ over $x^2 + y^2 + z^2=1$

2. Compute $\iint (a^2x^2 + b^2y^2 + c^2z^2)^{\frac{1}{2}}\, dS$ over the ellipsoid $a^2x^2 + b^2y^2 + c^2z^2=1$

(Hint: Volume of the ellipsoid , V= $\frac{4\pi}{3\sqrt{abc}}$)

3. Find $\int_s \bar{F}.\bar{n}dS$ where $\bar{F} = 2x^2\bar{i} - y^2\bar{j} + 4xz\bar{k}$ and S is the region in the first octant bounded by $y^2 + z^2=9$ and x=0,x=2.

4. Find $\int_s (4.x\bar{i} - 2y^2\bar{j} + z^2k).\bar{n}\, dS$ Where S Is the region bounded by $x^2 + y^2 =4$, z=0 and z=3.

5. Verity divergence theorem for F=6z$\bar{i}$ + (2x+y) $\bar{j}$ -x$\bar{k}$, taken over the region bounded by the surface of the cylinder $x^2 + y^2 =9$ included in z=0, z=8, x=0 and y=0. [JNTU 2007 S(Set No.2)]

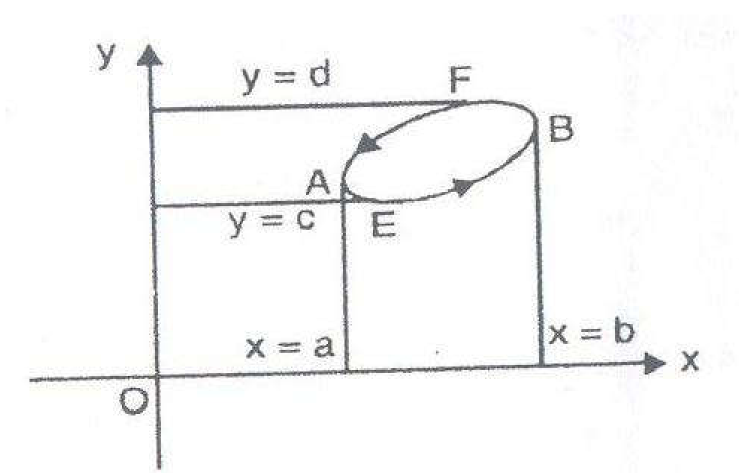
<u>II. GREEN'S THEOREM IN A PLANE</u>

(Transformation Between Line Integral and Surface Integral) [JNTU 2001S].

If S is Closed region in xy plane bounded by a simple closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R, then

$\oint_c M dx + N dy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy,$

Where C is traversed in the positive(anti clock-wise) direction



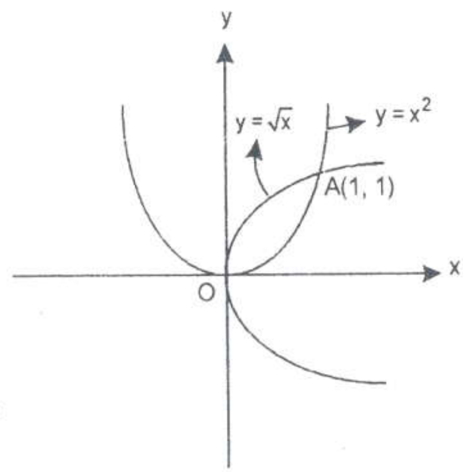
Solved Examples

Example 1: Verify Green's theorem in plane for

$\oint (3x^2 - 8y^2) dx + (4y - 6xy) dy$ where C is the region bounded by y=$\sqrt{x}$ and y=x^2 .

<u>**Solution**</u>: Let M=3x^2-8y^2 and N=4y-6xy. Then

$\frac{\partial M}{\partial y} = -16y, \frac{\partial N}{\partial x} = -6y$



We have by Green's theorem,

$\oint_c M dx + N dy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$

Now $\int_s \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy = \iint_s (16y - 6y) dx dy$

=10

$\iint_s y\, dx dy = 10 \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} y\, dy dx = 10 \int_{x=0}^{1} \left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$

=5

$\int_0^1 (x - x^4) dx = 5\left(\frac{x^2}{2} - \frac{x^5}{5}\right)_0^1 = 5\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{2}$ ….(1)

Verification:

We can write the line integral along c

=[line integral along y=x^2(from O to A) + [line integral along y^2 =x(from A to O)]

=I_1+I_2(say)

Now $I_1 = \int_{x=0}^{1} \{[3x^2 - 8(x^2)^2] dx + [4x^2 - 6x(x^2)] 2x dx\}$ $\left[\because y = x^2 \Rightarrow \frac{dy}{dx} = 2x\right]$

$$=\int_0^1 (3x^3 + 8x^3 - 20x^4)dx = -1$$

And

$$I_2 = \int_1^0 \left[(3x^2 - 8x)dx + \left(4\sqrt{x} - 6x^{3/2}\right)\frac{1}{2\sqrt{x}} = dx\right] = \int_1^0 (3x^2 - 11x + 2)dx = \frac{5}{2}$$

$\therefore I_1 + I_2 = -1 + 5/2 = 3/2.$

From(1) and (2), we have $\oint_c Mdx + Ndy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Hence the verification of the Green's theorem.

Example 2: Evaluate by Green's theorem $\int_c (y - \sin x)dx + \cos x\, dy$

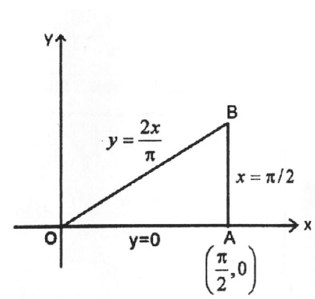
where C is the triangle enclosed by the lines y=0, x=$\frac{\pi}{2}$, $\pi y = 2x$.

[JNTU 1993, 1995 S, 2003 S, 2007, (H) June 2010(Set No.2)]

Solution : Let M=y-$\sin x$ $and\ N = \cos x$ Then

$\frac{\partial M}{\partial y}$=1 and $\frac{\partial N}{\partial x}$=-$\sin x$

By Green's theorem $\oint_c Mdx + Ndy = \int_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$



$$\Rightarrow \int_c (y - \sin x)dx + \cos x\, dy = \iint_s (-1 - \sin x)dx$$

$$=-\int_{x=0}^{\pi/2}\int_{y=0}^{\frac{2x}{\pi}}(1 +$$

$$=\int_{x=0}^{\pi/2}(\sin x + 1)$$

$$=\frac{-2}{\pi}\int_{x=0}^{\pi/2} x(\sin x + 1)dx$$

$$=$$

$$\frac{-2}{\pi}[(-\cos x + x)]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} 1(-\cos x + x)dx$$

$=$

$\frac{-2}{\pi}\left[x(-\cos x + x) + \sin x - \frac{x^2}{2}\right]_0^{\pi/2}$

$=$

$\frac{-2}{\pi}\left[-x\cos x + \frac{x^2}{2} + \sin x\right]_0^{\pi/2} = \frac{-2}{\pi}\left[\frac{\pi^2}{8} + 1\right] = -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$

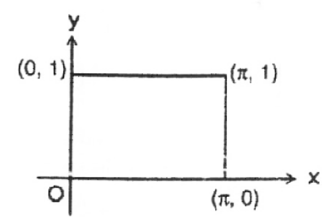
Example 3: Evaluate by Green's theorem for

$\oint_c (x^2 - \cosh y)dx + (y + \sin x)dy$ where C is the rectangle with vertices $(0,0), (\pi,0), (\pi,1), (0,1)$.

Solution: Let $M = x^2 - \cosh y, N = y + \sin x$

$$\therefore \frac{\partial M}{\partial y} = -\sinh y \ \ and \ \ \frac{\partial N}{\partial x} = \cos x$$

By Green's theorem, $\oint_c Mdx + Ndy = \int_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$



$$\Rightarrow \oint_c (x^2 - \cosh y)dx + (y + \sin x)dy = \int_s \int (\cos x + \sinh$$

$= \int_{x=0}^{\pi} \int_{y=0}^{1} (\cos x + \sinh y)dydx = \int_{x=0}^{\pi} (y\cos x + \cosh y)_0^1 dx$

$=$

$\int_{x=0}^{\pi} [\cos x + \cosh 1 - 1]dx = (\sin x + x\cosh 1 - x)_0^{\pi}$

$= \pi(\cosh 1 - 1)$

Example 4: A Vector field is given by $\bar{F} = (\sin y)\bar{i} + x(1 + \cos y)\bar{j}$.

Evaluate the line integral over the circular path $x^2 + y^2 = a^2$, z=0

(i) Directly (ii) By using Green's theorem

Solution : (i) Using the line integral

[JNTU 96, (A) June 2011 (Set No.4)

$$\oint_c \overline{F}.d\overline{r} = \oint_c F_1 dx + F_2 dy = \oint_c \sin y dx + x(1+\cos y)dy$$

$$= \oint_c \sin y dx + x \ \cos y \, dy = x dy = \oint_c d(x \sin y) + x dy$$

Given Circle is $x^2+y^2 = a^2$. Take x=a $\cos\theta$ and y=a $\sin\theta$ so that dx=-a $\sin\theta \, d\theta$ and

dy=a $\cos\theta \, d\theta$ and $\theta = 0 \to 2\pi$

$$\therefore \oint \overline{F}.d\overline{r} = \int_0^{2\pi} d[a \cos\theta \sin(a \sin\theta)] + \int_0^{2\pi} a(\cos\theta) a \cos\theta \, d\theta$$

$$= [a \cos\theta \sin(a \sin\theta)]_0^{2\pi} + 4a^2 \int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$= 0 + 4a^2.\frac{1}{2}.\frac{\pi}{2} - \pi a^2$$

(ii)Using Green's theorem

Let M=$\sin y$ and N=x$(1+\cos y)$. Then

$\frac{\partial M}{\partial y} = \cos y$ and $\frac{\partial N}{\partial x} = -(1+\cos y)$

By Green's theorem,

$$\oint_c M dx + N dy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$$

$$\therefore \oint_c \sin y \, dx + (1+\cos y)dy = \int_s \int (-\cos y + 1\cos y) dx dy == \iint dx dy$$

$$=$$

$$\int_s \int dA = A = \pi a^2 \ (\because area \ of \ circle = \pi a^2)$$

We observe that the values obtained in (i) and (ii) are same to that Green's theorem is verified.

Example 5: Show that area bounded by a simple closed curve C is given by $\frac{1}{2}\oint xdy - ydx$ and hence find the area of

(i)The ellipse x=$acos\theta \ [\![\theta, y = bsin\theta \ [\![\theta \ (i.e) \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1]\!] \ ]\!]$

(II)The Circle x=$a\cos\theta, y = a\sin\theta \ (i.e)x^2 + y^2 = a^2$

<u>Solution:</u> We have by Green's theorem

$\oint_c Mdx + Ndy = \int_s \int \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Here M=-y and N=x so that $\frac{\partial M}{\partial y} = -1 \ and \frac{\partial N}{\partial x} = 1$

$\oint_c xdy - ydx = 2 \int_s dxdy = 2A$ where A is the area of the surface.

$\therefore \frac{1}{2}\int xdy - ydx = A$

(i)For the ellipse x=$a\cos\theta$ and y=$b\sin\theta$ and $\theta = 0 \to 2\pi$

$\therefore Area, A = \frac{1}{2}\oint xdy - ydx = \frac{1}{2}\int_0^{2\pi}[(a \ \cos\theta)(b\cos\theta) - (b \ \sin\theta \ (-a \ \sin\theta))]d\theta$

$= \frac{1}{2}ab\int_0^{2\pi}(\cos^2\theta + sin^2\theta) \ d\theta = \frac{1}{2}ab(\theta)_0^{2\pi} = \frac{ab}{2}(2\pi - 0) = \pi ab$

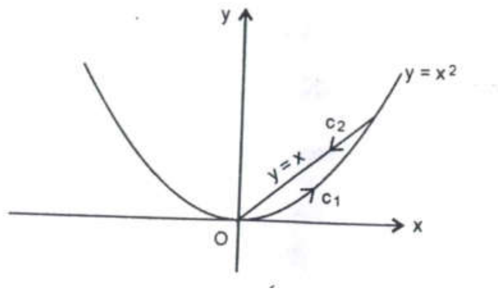
(ii)Put a=b to get area of the circle A=πa^2

Example 6: Verify Green's theorem for $\int_c[(xy + y^2)dx + x^2dy]$, where C is bounded by y=x and y=x^2

Solution:

By Green's theorem, we have $\int_c Mdx + Ndy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Here M=xy +y^2 and N=x^2



The line y=x and the parabola y=x^2 intersect at O$(0,0)$ and A$(1,1)$

Now $\int_c Mdx + Ndy = \int_{c_1} Mdx + Ndy + \int_{c_2} Mdx + Ndy$ …..(1)

Along c_1 $(i.e. \, y = x^2)$, the line integral is

$$\int_{c_1} Mdx + Ndy = \int_{c_1} [x(x^2) + x^4]dx + x^2 d(x^2) \int_c (x^3 + x^4 + 2x^3)dx = \int_0^1 (3x^3 + x^4)\, dx$$

$$= \left(3.\frac{x^4}{4} + \frac{x^5}{5}\right)_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

…….(2)

Along c_2 $(i.e. \, y = x)$ from $(1,1)$ to $(0,0)$, the line integral is

$$\int_{c_2} Mdx + Ndy = \int_{c_2} (x.x + x^2)dx + x^2 dx \; [\because dy = dx]$$

$$= \int_{c_2} 3x^2 dx = 3 \int_1^0 x^2 dx = 3\left(\frac{x^3}{3}\right)_1^0 = (x^3)_1^0 = 0-1=-1$$

….(3)

From (1), (2) and (3), we have

$$\int_c Mdx + Ndy = \frac{19}{20} - 1 = \frac{-1}{20}$$

…(4)

Now $\iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_s (2x - x - 2y)\,dxdy$

$=\int_{x=0}^{1}\int_{y=x^2}^{x}(x-2y)dxdy = \int_0^1 (xy - y^2)_{x^2}^{x}\,dx$

$=$

$\int_0^1[(x^2-x^2)-(x^3-x^4)]dx = \int_0^1(x^4-x^3)dx$

$=\left(\frac{x^5}{5}+\frac{x^4}{4}\right)_0^1 = \frac{1}{5}\quad\frac{1}{4} = \frac{-1}{20}$

….(5)

From $(4)\,and(5), We\ have \int_c Mdx + Ndy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

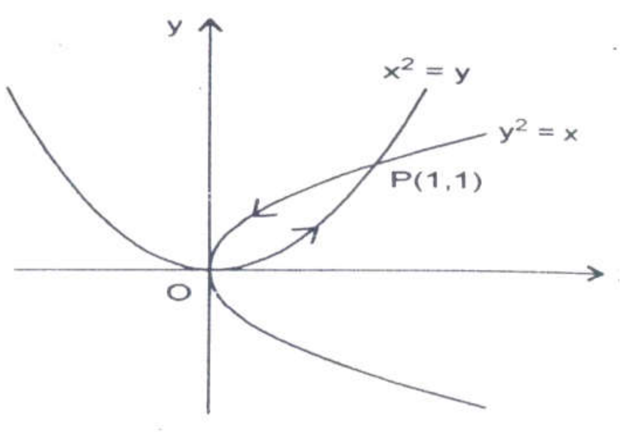
Hence the verification of the Green's theorem.

Example 7: Using Green's theorem evaluate

$\int_c (2xy - x^2)dx + (x^2+y^2)dy,$ Where "C" is the closed curve of the region bounded by y=x^2 and $y^2 = x$

[JNTU 2003S, 2006S,2008S, Aug2008S, June 2009, (K) Nov 2009 S (Set No.1)]

Solution:



The two parabolas $y^2 = x$ and $y = x^2$ are intersecting at O(0,0), and P(1,1)

Here M=2xy-x^2 and N=x^2 +y^2

$\therefore \frac{\partial M}{\partial y} = 2x$ and $\frac{\partial N}{\partial x} = 2x$

Hence $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - 2x = 0$

By Green's theorem $\int_C M dx + N dy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

i.e., $\int_C (2x - x^2) dx + (x^2 + y^2)\, dy = \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (0)\, dxdy = 0$

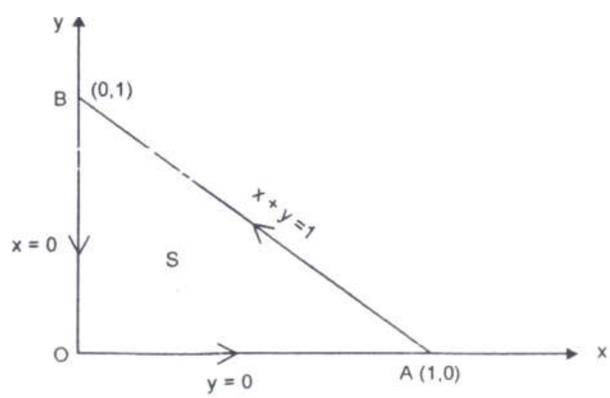
Example 8: Verify Green's theorem for

$\int_c [(3x^2 - 8y^2) dx + (4y - 6xy)\, dy]$ where c is the region bounded by x=0, y=0 and x+y=1. [JNTU 2003S, 2007S(Set No.3)

Solution : By Green's theorem, we have

$\int_C M dx + N dy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Here M=3$x^2 - 8y^2$ and N=4y-6xy



$\therefore \frac{\partial M}{\partial y} = 16y$ and $\frac{\partial N}{\partial x} = 6y$

Now $\int_C Mdx + Ndy = + \int_{OA} Mdx + Ndy + \int_{AB} Mdx + Ndy + \int_{BC} Mdx + Ndy$

….(1)

Along OA, y=0 $\therefore dy = 0$

$\int_{OA} Mdx + Ndy = \int_0^1 3x^2 dx = \left(\frac{x^3}{3}\right)_0^1 = 1$

Along AB, x+y=1 $\therefore dy = -dx$ and x=1-y and varies from 0 to 1.

$\int_{AB} Mdx + Ndy = \int_0^1 [3(y-1)^2 - 8y^2](-dy) + [4y + 6y(y-a)]dy$

$= \int_0^1 (-5y^2 - 6y + 3)(-dy) + (6y^2 - 2y)dy$

$= \int_0^1 (11y^2 + 4y - 3)dy = \left(11\frac{y^3}{3} + 4\frac{y^2}{2} - 3y\right)_0^1$

$= \frac{11}{3} + 2 - 3 = \frac{8}{3}$

Along BO, x=0 $\therefore dx = 0$ and limits of y are from 1 to 0

. $\int_{BO} Mdx + Ndy = \int_1^0 4ydy = \left(4\frac{y^2}{2}\right)_1^0 = (2y^2)_0^1 = -2$

from (1), we have $\int_C Mdx + Ndy = 1 + \frac{8}{3} - 2 = \frac{5}{3}$

Now $\iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_{x=0}^1 \int_{y=0}^{1-x} (-6y + 16y)dxdy$

$=10 \int_{x=0}^1 \left[\int_{y=0}^{1-x} ydy\right] dx = 10 \int_0^1 \left(\frac{y^2}{2}\right)_0^{1-x} dx$

$=5 \int_0^1 (1-x)^2 dx = 5\left[\frac{(1-x)^3}{-3}\right]_0^1$

$=-\frac{5}{3}[(1-1)^3 - (1-0)^3] = \frac{5}{3}$

From (2) and (3), we have $\int_C Mdx + Ndy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

Hence the verification of the Green's Theorem.

Example 9: Apply Green's theorem to evaluate

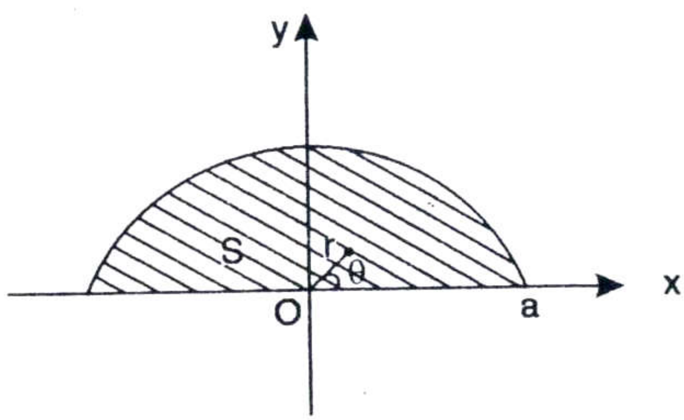
$\oint_c (2x^2 - y^2)dx + (x^2 + y^2)\,dy, where\ c\ is$

The boundary of the area enclosed by the x-axis and upper half of the circle $x^2 + y^2 = a^2$

[JNTU 2008S, (A)June 2010, 2011(Set No.2)]

Solution : Let M=$2x^2 - y^2$ and N=$x^2 + y^2$ Then

$\frac{\partial M}{\partial y} = -2y\ and\ \frac{\partial N}{\partial x} = 2x$



Figure

$\therefore\ By\ Green's Theorem, \int_c Mdx + Ndy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

$\Rightarrow \oint_c [(2x^2 - y^2)\,dx + (x^2 + y^2)dy] = \iint_s (2x + 2y)dxdy$

$=2\iint_s (x + y)dy$

$=2\int_0^a\int_0^{\pi} r(\cos\theta+\sin\theta).rd\theta dr$

[Changing to polar coordinates (r,θ), r varies from 0 to a and θ varies from 0 to π]

$$\therefore \oint_c [(2x^2-y^2)dx+(x^2+y^2)dy] = 2\int_0^a r^2\,dr \int_0^{\pi}\int_0^{\pi}(\cos\theta+\sin\theta)d\theta$$

=2.

$\frac{a^3}{3}(1+1)=\frac{4a^3}{3}$

Example 10: Find the area of the Follum of Descartes

$x^3+y^3=3axy(a>0) using\ Green's$

Theorem.

[JNTU 2006(Set No.1]

Solution: from Green's theorem, we have

$\int Pdx+Qdy=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$

By Green's theorem, Area = $\frac{1}{2}\oint_c(xdy-ydx)$

Considering the loop of follum Descartes(a>0)

Let x=$\frac{3at}{1+t^3}, y=\frac{3at^2}{1+t^3}$. Then dx=$\left[\frac{d}{dt}\left(\frac{3at}{1+t^3}\right)\right]dt$ and dy=$\left[\frac{d}{dt}\left(\frac{3at^2}{1+t^3}\right)\right]dt$

The point of intersection of the loop is $\left(\frac{3a}{2}-\frac{3a}{2}\right)\Rightarrow t=1$

Along OA, t varies from 0 to1.

$$\therefore \frac{1}{2}\oint(xdy-ydx)=\frac{1}{2}\int_0^1\left(\frac{3at}{1+t^3}\right)\left[\frac{d}{dt}\left(\frac{3at^2}{1+t^3}\right)\right]dt-\left(\frac{3at^2}{1+t^3}\right)\left[\frac{d}{dt}\left(\frac{3at}{1+t^3}\right)\right]dt$$

$$=\frac{1}{2}\int_0^1 \left\{\frac{3at}{1+t^3}\left[\frac{3at(2-t^3)}{(1+t^3)^2}\right]-\frac{3at^2}{1+t^3}\left[\frac{3a(1-2t^3)}{(1+t^3)^2}\right]\right\} dt$$

$$=\frac{9a^2}{2}\int_0^1 \left[\frac{t^2(2-t^3)}{(1+t^3)^3}-\frac{t^2(1-2t^3)}{(1+t^3)^3}\right] dt = \frac{9a^2}{2}\int_0^1 \frac{2t^2-t^5-t^2+2t^5}{(1+t^3)^3}\, dt$$

$$=\frac{9a^2}{2}\int_0^1 \frac{t^2+t^5}{(1+t^3)^3}\, dt = \frac{9a^2}{2}\int_0^1 \frac{t^2(1-t^3)}{(1+t^3)^3}\, dt$$

$=\frac{9a^2}{2}\int_0^1 \frac{t^2}{(1+t^3)^2}\, dt$ [Put $1+t^3 = x \Rightarrow 3t^2\, dt = dx$ L.L. : x=1, U.L.:x=2]

$=\frac{9a^2}{2}\int_{10}^2 \frac{t^2}{x^2}\cdot\frac{dx}{3t^2} = \frac{9a^2}{6}\int_{10}^2 \frac{1}{x^2}\, dx = \frac{3a^2}{4}\, sq.$ units(a>0).

Example 11: Verify Green's theorem in the plane for

$$\int_c (x^2 - xy^3)\, dx + (y^2 - 2xy)dy$$

Where C is square with vertices (0,0), (2,0), (2,2), (0,2).

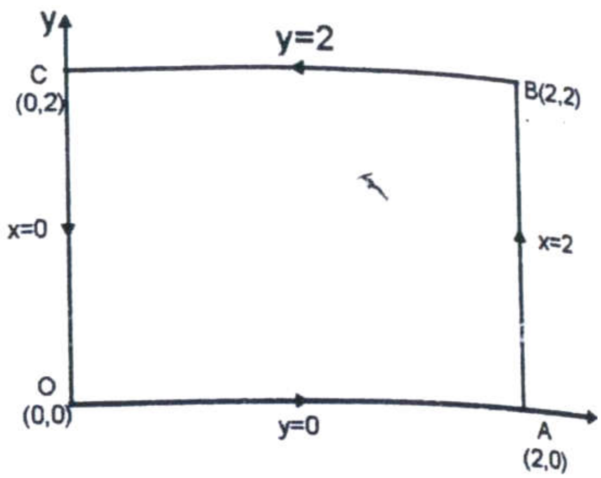
[JNTU Aug, 2008S, (H)June2009,(K) May2010(Set No.2)]

Solution: The Cartesian form of Green's theorem in the plane is

$$\int_C M dx + N dy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$$

Here M=$x^2 - xy^3$ and N=$y^2 - 2xy$

$\therefore\ \frac{\partial M}{\partial y}$=3 xy^2 and $\frac{\partial N}{\partial x} = -2y$



Evaluation of $\int_c (M\,dx + N\,dy)$

To Evaluate $\int_c (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$, we shall take C in four different segments viz (i) along OA(y=0) (ii) along AB(x=2) (iii) along BC(y=2) (iv) along CO(x=0).

(i)Along OA(y=0)

$\int_c (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_0^2 x^2 dx = \left(\frac{x^3}{3}\right)_0^2 = \frac{8}{3}$

…..(1)

(ii)Along AB(x=2)

$\int_c (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy = \int_0^2 (y^2 - 4y)\,dy \quad [\because x = 2, dx = 0]$

=

$\left(\frac{y^3}{3} - 2y^2\right)_0^2 = \left(\frac{8}{3} - 8\right) = 8\left(-\frac{2}{3}\right) = -\frac{16}{3}$ ….(2)

(iii)Along BC(y=2)

$\int_c (x^2 - xy^3)\,dx + (y^2 - 2xy)dy = \int_2^0 (x^2 - 8x)\,dx \quad [\because y = 2, dy = 0]$

$$=\left(\frac{x^3}{3}-4x^2\right)_2^0=\left(\frac{8}{3}-16\right)=\frac{40}{3}$$

…(3)

(iv)Along CO(x=0)

$$\int_C (x^2-xy^3)\,dx+(y^2-2xy)dy=\int_2^0 y^2\,dx\ \ [\because x=0, dx=0]=\left(\frac{y^3}{3}\right)_2^0=-\frac{8}{3}$$

…..(4)

Adding(1),(2),(3) and (4), we get

$$\int_C (x^2 \quad xy^3)\,dx + (y^2 \quad 2xy)dy=\frac{8}{3} \quad \frac{16}{3}+\frac{40}{4} \quad \frac{8}{3}=\frac{24}{3}=8$$

…(5)

Evaluation of $\iint_s \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) \boldsymbol{dxdy}$

Here x ranges from 0 to 2 and y ranges from 0 to 2.

$$\iint_s \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dxdy=\int_0^2\int_0^2(-2y+3xy^2)dxdy$$

$$=\int_0^2\left(-2xy+\frac{3x^2}{2}y^2\right)_0^2 dy$$

$$=\int_0^2(-4y+6y^2)\,dy=(-2y^2+2y^3)_0^2$$

$$=-8+16=8$$

…(6)

From (5) and (6), we have

$$\int_C Mdx+Ndy=\iint_s \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) dxdy$$

Hence the Green’s theorem is verified.

Assignments

(1) Evaluate $\oint_c (3x+4y)dx+(2x-3y)\,dy$ where c is the circle $x^2+y^2=4$

(2) Verify Green's theorem in the plane for $\oint (x^2-xy^3)dx+(y^2-2xy)dy$ where c is the square with vertices (0,0), (2,0), (2,2) and (0,2). [JNTU Sep 2008, 2008S, JNTU(H) 2009(Set No.1)]

(3) Use Green's theorem to evaluate $\oint_c x^2(1+y)\,dx+(y^3+x^3)\,dy$ where c is the square bounded by y=± 1 and $x=\pm 1$.

(4) Find the area bounded by one arc of the cycloid x=a $(\theta-\sin\theta), y=a(1-\cos\theta), a>0$ and the $x-axis.$

(5) Find the area bounded by the hypocycloid $x^{2/3}+y^{2/3}=a^{2/3}, a>0.$

(6) Find $\oint (x^2-y^2)dx+dx+3xy^2dy$ where c is the circle $x^2+y^2=4$ in xy plane.

Answers

(1)-8π (3)$\frac{8}{3}$ (4)3πa^2 (5)$\frac{3\pi a^2}{8}$ (6)12π

III. STOKE'S THEOREM

(Transformation between Line Integral and Surface Integral) [JNTU 2000]

Let S be a open surface bounded by a closed, non intersecting curve G. if $\bar{F}$ is any differentieable vector point function then $\oint_c \bar{F}.d\,\bar{r}=\int_s curl\ \ \bar{F}.\bar{n}\,ds$ *where c is traversed in the positive* direction and $\bar{n}$ *is unit outward drawn normal at any point of the surface.*

Solved Examples

Example 1: Prove by Stokes theorem, Curl grad $\phi=\bar{0}$

Solution: Let S be the surface enclosed by a simple closed curve C.

$\therefore$ *By Stokes theorem*

$$\int_s (curl\ grand\phi).\bar{n}\ ds = \int_s (\nabla x \nabla \phi).\bar{n}\ dS = \oint_C \nabla\phi.d\bar{r} = \oint_C \nabla\phi.d\bar{r}$$

$$=\oint_C\left(\frac{i\partial\phi}{\partial x}+\bar{j}\frac{\partial\phi}{\partial y}+\bar{k}\frac{\partial\phi}{\partial z}\right)-(\bar{\imath}dx+\bar{j}dy+\bar{k}dz)$$

$$=\oint_C\left(\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz\right)=\int d\phi=\phi]_p$$

where P is any point on C.

$$\int curl\ grad\,\phi.\bar{n}\ ds = \bar{0} \Rightarrow curl\ grad\phi = \bar{0}$$

Example 2: prove that $\int_s \phi curl\,\bar{f}.dS = \int_c \phi\,\bar{f}.d\bar{r} - \int_s curl\ grad\phi\ X\ \bar{f}dS$

Solution: Applying Stokes theorem to the function $\phi\,\bar{f}$

$$\int_C \phi\,\bar{f}.d\bar{r} = \int curl(\phi\bar{f}).\bar{n}\,ds = \int_s (grad\phi\ X\ \bar{f} + \phi curl\bar{f})\ ds$$

$\therefore \int_s \phi \, curl \bar{f} . ds = \int_C \phi \, \bar{f} . d\bar{r} - \int \nabla\phi \, X \, \bar{f} . ds$

Example 3: Prove that $\oint_c f\nabla f . d\bar{r} = 0.$

Solution: By Stokes Theorem,

$\oint_c (f\nabla f) . d\bar{r} = \int_s curl f \nabla f . \bar{n} \, ds = \int_s [f \, curl \nabla f + \nabla f X \, \nabla f] . \bar{n} \, ds$

$\int \bar{0} . \, \bar{n} ds = 0. [\because \; curl \nabla f = \bar{0} \; and \; \nabla f X \, \nabla f = \bar{0}]$

Example 4: Prove that $\oint_c f\nabla g . d\bar{r} = \int (\nabla f X \, \nabla g) . \bar{n} ds$

Solution: By Stokes Theorem,

$\oint_c (f\nabla g . d\bar{r}) = \int_s [\nabla X (f\nabla g)] . \bar{n} \, ds = \int_s [\nabla f \, X \, \nabla g + f \;\; curl \, grad \, g] . \bar{n} ds$

$= \int [\nabla X (f\nabla g)] . \bar{n} \, ds \quad [\because curl(grad \, g) = \bar{0}]$

Example 5: Verify Stokes theorem for $\bar{F} = -y^3 \bar{i} + x^3 \bar{j}$, Where S is the circular disc

$x^2 + y^2 \leq 1, z = 0.$ [JNTU 99,2007,2008S(Set No.4)]

Solution: Given that $\bar{F} = -y^3 \bar{i} + x^3 \bar{j}$. The boundary of C of S is a circle in xy plane.

$x^2 + y^2 \leq 1, z = 0.$ We use the parametric co-ordinates x=cos $\theta, y = sin\theta, z = 0, 0 \leq \theta \leq 2\pi;$

dx=-sinθ $d\theta$ and dy =cosθ $d\theta$

$\therefore \oint_c \bar{F}.dr = \int_c F_1 dx + F_2 dy + F_3 dz = \int_c -y^3 dx + x^3 dy$

$$= \int_0^{2\pi} [-sin^3\theta(-sin\theta) + cos^3\theta cos\theta] d\theta = \int_0^{2\pi} (cos^4\theta + sin^4\theta) d\theta$$

$$= \int_0^{2\pi} (1 - 2sin^2\theta \; cos^2\theta) d\theta = \int_0^{2\pi} d\theta \quad \frac{1}{2}\int_0^{2\pi} (2sin\theta \; cos\theta)^2 \, d\theta$$

$$= \int_0^{2\pi} d\theta - \frac{1}{2}\int_0^{2\pi} sin^2 2d\theta = (2\pi - 0) - \frac{1}{4}\int_0^{2\pi} (1 - cos4\theta) d\theta$$

$$= 2\pi + \left[-\frac{1}{4}\theta + \frac{1}{16} sin4\theta\right]_0^{2\pi} = 2\pi - \frac{2\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$$

Now $\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = \bar{k}(3x^2 + 3y^2)$

$\therefore \int_s (\nabla \times \bar{F}).\bar{n}ds = 3\int_s (x^2 + y^2)\bar{k}.\bar{n}ds$

We have $(\bar{k}.\bar{n})ds - dx \; dy \; and \; R \; is \; the \; region \; on \; xy - plane$

$\therefore \iint_s (\nabla \times \bar{F}).\bar{n}ds = 3\iint_R (x^2 + y^2) \; dx \; dy$

Put x=r cos$\emptyset$, $y = r \; sin\emptyset \therefore dxdy = rdr \; d\emptyset$

R is varying from 0 to 1 and 0$\leq \emptyset \leq 2\pi$.

$\int(\nabla \times \bar{F}).\bar{n}ds = 3\int_{\emptyset=0}^{2\pi}\int_{r=0}^{1} r^2$.rdr d$\emptyset - \frac{3\pi}{2}$

L.H.S=R.H.S.Hence the theorem is verified.

Example6: If $\bar{F} = y\bar{i} + (x - 2xz)\bar{j} - xy\bar{k}$, $evaluate \; \int_s (\nabla \times \bar{F}).\bar{n}ds \; Where$ Sis the surface of sphere $x^2 + y^2 + z^2 = a^2$, $above \; the \; xy - plane.$

<u>Solution:</u> Given$\bar{F} = y\bar{i} + (x - 2xz)\bar{j} - xy\bar{k}$.

By Stokes Theorem,

$\int_s (\nabla \times \bar{F}).\bar{n}ds = \int_c F_1 dx + F_2 dy + F_3 dz = \int_c ydx + (x - 2xz)dy - xydz$

Above the xy plane the sphere is $x^2 + y^2 += a^2, z = 0$

$\therefore \int_c \bar{F}.d\bar{r} = \int_c ydx + xdy.$

Put x=a cos θ,y=asinθ so that $dx = -a\ sin\theta d\theta, dy = acos\theta d\theta\ and \theta = 0 \rightarrow 2\pi$

$$\int_c \bar{F}.d\bar{r} = \int_0^{2\pi} (a\ sin\theta)(-a\ sin\theta)\ d\theta + (acos\theta)(acos\theta)d\theta$$

$$= a^2 \int_0^{2\pi} cos2\theta\ d\theta = a^2 \left[\frac{sin2\theta}{2}\right]_0^{2\pi} = \frac{a^2}{2}(0) = 0$$

Example 7: Verify Stokes theorem for $\bar{F} = (2x - y)\bar{\imath} - yz^2\bar{\jmath} - y^2z\bar{k}$ over the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$bounded by the projection of the xy-plane.

[JNTU2006,2007,2007S,2008,JNTU(A) June2009(Set No.2)]

Solution: The boundary C of S is a circle in xy plane i.e $x^2 + y^2 = 1$, z=0

The parametric equations are x=$cos\theta, y = sin\theta, \theta = 0 \rightarrow 2\pi$

$\therefore\ dx = -sin\theta\ d\theta, dy = cos\theta\ d\theta$

$\int_C \bar{F}.d\bar{r} = \int_C \bar{F}.d\bar{r} = \int_c \bar{F}_1.dx + \bar{F}_2.dy + \bar{F}_3.dz = \int_c (2x - y)dx - yz^2dy - y^2zdz$

$$= \int_c (2x - y)dx(since\ z = 0\ and\ dz = 0)$$

$$= \int_c^{2\pi} (2cos\theta - sin\theta)sin\theta\ d\theta = \int_c^{2\pi} sin^2\theta\ d\theta = \int_c^{2\pi} sin2\ d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{1-cos2\theta}{2} d\theta - \int_0^{2\pi} sin2\theta\ d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}sin2\theta + \frac{1}{2}.cos2\theta\right]_0^{2\pi}$$

$$=\frac{1}{2}(2\pi - 0) + 0 + \frac{1}{2}.(cos4\pi - cos0) = \pi$$

Again $\nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = \bar{\imath}(-2yz + 2yz) - \bar{\jmath}(0-0) + \bar{k}(0+1) = \bar{k}$

$\therefore \int_S (\nabla \times \bar{F}).\bar{n}ds = \int_S \bar{k}.\bar{n}ds = \int_R \int dxdy$

Where R is the projection of S on xy plane and $\bar{k}.\bar{n}ds = dxdy$

Now

$$\int\int_R dxdy = 4\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} dydx = 4\int_{x=0}^{1}\sqrt{1-x^2}\,dx = 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]_0^1$$

$$=4\left[\frac{1}{2}\sin^{-1}1\right] = 2\frac{\pi}{2} = \pi$$

$\therefore$ *The* Stokes theorem is verified.

Example 8: Verify Stokes theorem for the function $\bar{F} = x^2\,\bar{\imath} + xy\,\bar{\jmath}$ integrated round the square in the plan z=0 whose sides are along the lines x=0, y=0, x=a, y=a.

Solution: Given $\bar{F} = x^2\,\bar{\imath} + xy\,\bar{\jmath}$

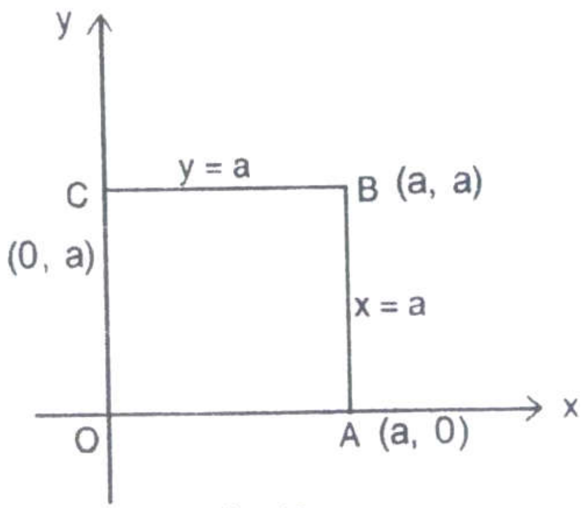


Fig. 13

By Stokes Theorem, $\int_S (\nabla \times \bar{F}).\bar{n}ds = \int_S \bar{F}.d\,\bar{r}$

Now $\nabla \times \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix} = \bar{k}y$

L.H.S. , $\int_S (\nabla \times \bar{F}).\bar{n}ds = \int_S y(\bar{n}.\bar{k})\,ds = \int_S ydxdy$

$n.\bar{k}.ds = dxdy$ and R is the region bounded for the square.

$\therefore \int_S (\nabla \times \bar{F}).\bar{n}ds = \int_0^a \int_0^a ydydx = \frac{a^3}{2}$

R.H.S. $= \int_C \bar{F}.d\bar{r} = \int_C (x^2dx + xydy)$

But $\int \bar{F}.d\bar{r} = \int_{OA} \bar{F}.d\bar{r} + \int_{AB} \bar{F}.d\bar{r} + \int_{BC} \bar{F}.d\bar{r} + \int_{CO} \bar{F}.d\bar{r}$

(i)Along OA: y=0, z=0, dy=0, dx=0

$\therefore \int_{OA} \bar{F}.d\bar{r} = \int_0^a x^2dx = \frac{a^3}{3}$

(ii)Along AB:x=a, z=0,dx=0,dz=0

$\int_{AB} \bar{F}.d\bar{r} = \int_0^a aydx = \frac{1}{2}a^3$

(iii)Along BC: y=a,z=0,dy=0,dz=0

$\therefore \int_{BC} \bar{F} \cdot d\bar{r} = \int_a^0 0 dx = \frac{1}{3} a^3$

(iv)Along CO: x=0, z=0, dx=0, dz=0

$\therefore \int_{CO} \bar{F} \cdot d\bar{r} = \int_a^0 0 dy = 0$

Adding $\int_c \bar{F} \cdot d\bar{r} = \frac{1}{3}a^3 + \frac{1}{2}a^3 + \frac{1}{3}a^3 + 0 = \frac{1}{2}a^3$

Hence the verification.

Example 9: Apply Stokes theorem, to evaluate $\oint_c (ydx + zdy + xdz)$ where c is the curve of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and x+z=a. [JNTU 1997, 2006S, 2008S(Set No.1,3)]

Solution : The intersection of the sphere $x^2 + y^2 + z^2 = a^2$ *and* the plane x+z=a. is a circle in the plane x+z=a. with AB as diameter.

Equation of the plane is x+z=a$\Rightarrow \frac{x}{a} + \frac{z}{a} = 1$

$\therefore OA + OB = a \ i.e. A = (a, 0,0) and \ B = (0,0, a)$

$\therefore length\ of\ the\ diameter\ AB = \sqrt{a^2 + a^2 + 0} = a\sqrt{2}$

Radius of the circle, r=$\frac{a}{\sqrt{2}}$

Let

$\bar{F}.d\bar{r} = ydx + zdy + xdz \Rightarrow \bar{F}.d\bar{r} = \bar{F}.(\bar{\imath}dx + \bar{\jmath}dy + \bar{k}dz) = ydx + zdy + xdz$

$\Rightarrow \bar{F} = y\,\bar{\imath} + z\,\bar{\jmath} + x\,\bar{k}$

$$\therefore curl\ \bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -(\bar{\imath} + \bar{\jmath} + \bar{k})$$

Let $\bar{n}$ be the unit normal to this surface. $\bar{n} = \frac{\nabla s}{|\nabla s|}$

Then s=x+z-a, $\nabla_s = \bar{i} + \bar{k}$. $\therefore$ $\bar{n} = \frac{\nabla s}{|\nabla s|} = \frac{\bar{i}+\bar{k}}{\sqrt{2}}$

Hence $\oint_c \bar{F}.d\bar{r} = \int curl\ \bar{F}.\bar{n}\, ds$ (by Stokes Theorem)

$$=-\int(\bar{i}+\bar{j}+\bar{k}).\left(\frac{\bar{i}+\bar{k}}{\sqrt{2}}\right)ds = -\int\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)ds$$

$$=-\sqrt{2}\int_s ds = -\sqrt{2}S = -\sqrt{2}\left(\frac{\pi a^2}{2}\right) = \frac{\pi a^2}{\sqrt{2}}$$

Example 10: Apply the Stoke's theorem and show that

$\int_s\int curl\ \bar{F}.\bar{n}d\bar{s} = 0$ where $\bar{F}$ is any vector and S $= x^2+y^2+z^2=1$

[JNTU 1998]

Solution: Cut the surface if the Sphere $x^2+y^2+z^2 = 1$ by any plane,

Let S_1 and S_2 denotes its upper and lower portions a C, be the common curve bounding both these portions.

$\therefore \int_S curl\ \bar{F}.d\bar{s} = \int_{S_1} \bar{F}.d\bar{s} + \int_{S_2} \bar{F}.d\bar{s}$

Applying Stoke's theorem,

$\int_S curl\ \bar{F}.d\bar{s} = \int_{S_1} \bar{F}.d\bar{R} - \int_{S_2} \bar{F}.d\bar{R} = 0$

The 2nd integral curl $\bar{F}.d\bar{s}$ is negative because it is traversed in opposite direction to first integral.

The above result is true for any closed surface S.

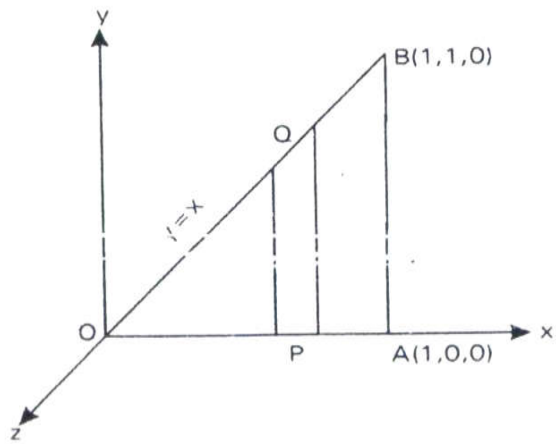
Example 11: Evaluate by Stokes theorem

$\oint_c (x+y)dx + (2x-z)dy + (y+z)dz$ where C is the boundary of the triangle with vertices (0,0,0), (1,1,0).

Solution: Let $\bar{F}.d\bar{r} = \bar{F}.(\bar{\imath}dx + \bar{\jmath}dy + \bar{k}dz) = (x+y)dx + (2x-z)dy + (y+z)dz$

Then $\bar{F} = (x+y)\bar{\imath} + (2x-z)\bar{\jmath} + (y+z)\bar{k}$

By Stokes theorem, $\oint_c \bar{F}.\,d\bar{r} = \int\int_s curl\,\bar{F}.\bar{n}\,ds$



Where S is the surface of the triangle OAB which lies In the xy plane. Since the z Co-ordinates of O,A and B Are zero. Therefore $\bar{n} = \bar{k}$. Equation of OA is y=0 and that of OB, y=x in the xy plane.

$$\therefore\ curl\,\bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2\bar{\imath} + \bar{k}$$

$\therefore\ curl\,\bar{F}.\bar{n}ds = curl\,\bar{F}.\bar{K}\,dx\,dy = dx\,dy$

$\oint_c \bar{F}.d\bar{r} = \int\int_s dx\,dy = \int\int_s dA = A = area\ of$ the $\Delta\,OAB$

$= \frac{1}{2}OA \times AB = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

Example 12: Use Stoke's theorem to evaluate $\int\int_S curl\ \bar{F}.\bar{n}\,dS$ over the surface if the paraboloid z+$x^2 + y^2 = 1, z \geq 0$ where

$\bar{F} = y\,\bar{i} + z\,\bar{j} + x\,\bar{k}.$

Solution : By Stoke's theorem

$\int_S curl\ \bar{F}.ds = \oint_c \bar{F}.d\bar{r} = \int_C (y\,\bar{i} + z\,\bar{j} + x\,\bar{k}).(\bar{i}dx + \bar{j}dy + \bar{k}dz)$

$= \int_S ydx\ (Since\ z = 0, dz = 0)$ …(1)

Where C is the circle $x^2 + y^2 = 1$

The parametric equations of the circle are x=$cos\theta, y = sin\theta$

$\therefore dx = -\ sin\theta\ d\theta$

Hence (1) becomes

$\int_S curl\ \bar{F}.ds = \int_{\theta=0}^{2\pi} sin\theta(-sin\theta)d\theta = -\int_{\theta=0}^{2\pi} sin^2\theta d\theta = -4\int_0^{\pi/2} sin^2\theta d\theta = -4\ X \frac{1}{2} X \frac{\pi}{2}$=-$\pi$

Example 13: Verify Stoke's theorem for $\bar{F} = (x^2 + y^2)\bar{i} - 2xy\bar{j}$ taken round the rectangle bounded by the lines x=$\pm a, y = 0, y = b.$

{JNTU 2003, 2005 (Set No.1)]

Solution: Let ABCD be the rectangle whose vertices are (a,0), (a,b), (-a,b) and (-a,0).

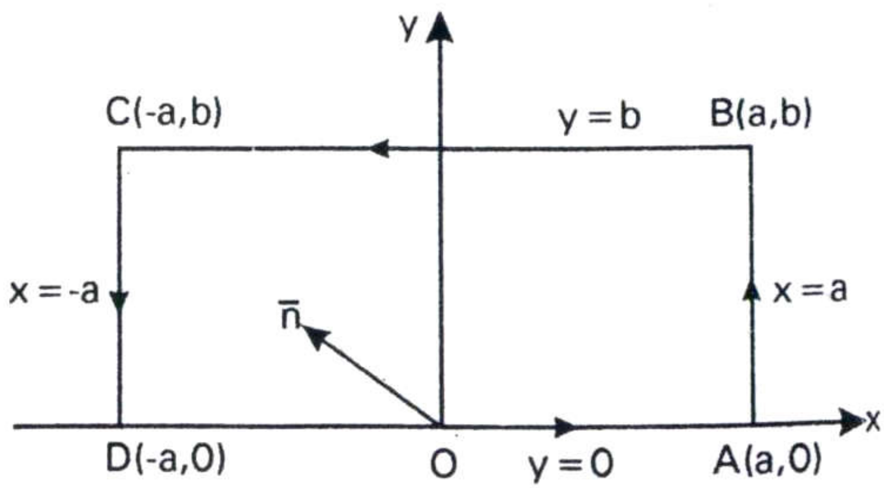
Equations of AB, BC, Cd and Da are x=a, y=b, x=-a and y=0.

We have to prove that $\oint_c \bar{F}.d\bar{r} = \int_S curl\ \bar{F}.\bar{n}ds$

$$\oint_c \bar{F}\cdot d\bar{r} = \oint_c \{(x^2+y^2)\bar{\imath} - 2xy\bar{\jmath}\}.\{\bar{\imath}dx + \bar{\jmath}dy\}$$

$$= \oint_c (x^2+y^2)\, dx - 2xydy$$

$$= \int_{AB} + \int_{Bc} + \int_{CD} + \int_{DA} \qquad \text{.....(1)}$$



(i)along AB, x=a, dx=0

from (1), $\int_{AB} = \int_{y=0}^{b} -2ay\, dy = -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2$

(ii)Along BC, y=b, dy=0

from (1), $\int_{BC} = \int_{x=a}^{x=-a}(x^2+b^2)dx \left[\frac{x^3}{3}b^2x\right]_{x=a}^{-a} = \frac{-2a^3}{3} - 2ab^2$

(iii)along CD, x=-a, dx=0

from (1), $\int_{CD} = \int_{y=b}^{0} 2ay\, dy = 2a[y^2]_{y=b}^{0} = -ab^2$

(iv)Along DA, y=0, dy=0

from (1), $\int_{DA} = \int_{x=-a}^{x=a} x^2dx \left[\frac{x^3}{3}\right]_{x=-a}^{a} = \frac{2a^3}{3}$

(i)+(ii)+(iii)+(iv) gives

$\therefore \oint_c \bar{F}.d\bar{r} = \quad ab^2 - -\frac{-2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2$

….(2)

Consider $\int_s curl\ \bar{F}.\bar{n}\, dS$

Vector Perpendicular to the xy-plane is $\bar{n} = k$

$$\therefore \ curl\ \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 + y^2) & -2xy & 0 \end{vmatrix} = 4y\bar{k}$$

Since the rectangle lies in the xy plane,

$\bar{n} = \bar{k}$ and ds =dx dy

$$\int_s curl\ \bar{F}.\bar{n} dS = \int_s -4y\bar{k}.\bar{k} dx\, dy = \int_{x=-a}^{a} \int_{y=0}^{b} -4y\, dx\, dy$$

$$= \int_{y=0}^{b} \int_{x=-a}^{a} -4y\, dx\, dy - 4\int_{y=0}^{b} y[x]_{-a}^{a} = -4\int_{y=0}^{b} 2ay\, dy$$

$$= \quad 4a[y^2]_{y=0}^{b} = \quad 4ab^2$$

…..(3)

Hence from (2) and (3), the Stoke’s theorem is verified.

Example 14: Verify Stoke’s theorem for

$\bar{F} = (y - z + 2)\bar{i} + (yz + 4)\bar{j} - xz\bar{k}$ where S is the surface of the cube x =0, y=0, z=0, x=2, y=2,z=2 above the xy plane.

{JNTU 2006S(Set No.1)]

Solution: Given $\bar{F} = (y - z + 2)\bar{i} + (yz + 4)\bar{j} - xz\bar{k}$ where S is the surface of the cube.

X=0, y=0, z=0, x=2, y=2, z=2 above the xy plane.

By Stoke's theorem, we have $\int curl\, F.\bar{n}ds = \int\ F.d\bar{r}$

$$\nabla\times\bar{F}=\begin{vmatrix}\bar{\imath} & \bar{\jmath} & \bar{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y-z+2 & y+4 & -xz\end{vmatrix}=\bar{\imath}(0+y)-\bar{\jmath}(-z+1)+\bar{k}(0-1)=y\bar{\imath}-(1-z)\bar{\jmath}-\bar{k}$$

$\therefore\ \nabla\times\bar{F}.\bar{n}=\nabla\times\bar{F}.k=\left(y\bar{\imath}-(1-z)\bar{\jmath}-\bar{k}\right).k=-1$

$\therefore\ \int\nabla\times\bar{F}.\bar{n}.ds=\int_0^2\int_0^2-1\ dx\ dy\ \ (\because z=0, dz=0)=-4$

…..(1)

To find $\int \bar{F}.d\bar{r}$

$\int\bar{F}.d\bar{r}=\int\left((y-z+2)\bar{\imath}+(yz+4)\bar{\jmath}-xz\bar{k}\right).\ (dx\imath+dy\jmath+dz\bar{k})$

$= \int[(y-z+2)dx+(yz+4)dy-(xz)dz]$

Sis the surface of the cube above the xy-plane

$\therefore z=0\ \Rightarrow dz=0$

$\therefore\ \int\bar{F}.d\bar{r}=\int(y+2)dx+\int 4dy$

Along $\overline{OA}, y=0, z=0, dy=0, dz=0, x\ change\ from\ 0\ to\ 2.$

$\int_0^2 2dx=2[x]_0^2=4$ ……..(2)

Along $\overline{BC}, y=2, z=0, dy=0, dz=0, x\ change\ from\ 2\ to\ 0.$

$\int_2^0 4dx=4[x]_2^0=-8$ ……. .(3)

Along $\overline{AB}, x=2, z=0, dx=0, dz=0, y\ change\ from\ 0\ to\ 2.$

$\int\bar{F}.d\bar{r}=\int_2^0 4dy=[4y]_0^2=8$ ……(4)

Along $\overline{CO}, x=0, z=0, dx=0, dz=0, y\ change\ from\ 2\ to\ 0.$

$.\int_2^0 4dy=-8$ …..(5)

Above the surface When z=2

Along $O'A'$, $\int_0^2 \bar{F}\,\overline{dr} = 0$ ….(6)

Along $A'B'$, $x = 2, z = 2, dx = 0, dz = 0$, y changes from 0 to 2

$\int_0^2 \bar{F}.d\bar{r} = \int_0^2 (2y+4)dy = 2.\frac{y^2}{2}\Big]_0^2 + 4y]_0^2 = 4 + 8 = 12$ ….(7)

Along $B'C'$, $y = 2, z = 2, dy = 0, dz = 0$, x changes from 2 to 0

$\int_0^2 \bar{F}.\overline{dr} = 0$

….(8)

Along $C'D'$, $x = 0, z = 2, dx = 0, dz = 0$, y changes from 2 to 0.

$\int_0^2 (2y+4) = 2.\frac{y^2}{2}\Big]_0^2 + 4y]_0^2 = -12$

…..(9)

(2)+(3)+(4)+(5)+(6)+(7)+(8)+(9) gives

$\int_c \bar{F}.d\bar{r} = 4 - 8 + 8 - 8 + 0 + 12 + 0 - 12 = -4$

…..(10)

By Stokes theorem, We have

$\int \bar{F}.d\bar{r} = \int curl\ \bar{F}.\bar{n}ds = -4$

Hence Stoke's theorem is verified.

Example 15: Verify the Stoke's theorem for $\bar{F} = y\bar{i} + z\bar{j} + x\bar{k}$ and surface is the part of the sphere $x^2 + y^2 + z^2 = 1$ *above the xy plane.*

Solution: Given $\bar{F} = y\bar{\imath} + z\bar{\jmath} + x\bar{k}$ over the surface

$x^2 + y^2 + z^2 = 1$ *is xy plane.*

We have to prove $\int_C \bar{F}.d\bar{r} = \int\int_s Curl\ \bar{F}.\bar{n}ds$

$\bar{F}.d\,\bar{r} = .(y\bar{\imath} + z\bar{\jmath} + x\bar{k}).(dx\bar{\imath} + dy\bar{\jmath} + dz\bar{k}) = ydx + zdy + xdz$

$\int_C (ydx + zdy + xdz) = \int ydx \quad (in\ xy\ plane\ z = 0, dz = 0)$

Let $x = cos\theta, y = sin\theta \Rightarrow dx = -sin\theta\ d\theta, dy = cos\theta\ d\theta$

$\therefore \int_C \bar{F}.d\bar{r} = \int_C y.dx = \int_0^{2\pi} ydx \qquad [\because x^2 + y^2 = 1, z = 0]$

$$= \int_0^{2\pi} sin\theta\ (-sin\theta)d\theta = -4\int_0^{\pi/2} sin^2\theta\ d\theta$$

$$= -4\int_0^{\pi/2} \frac{1-cos2\theta}{2} d\theta = -4\left[\left(\frac{1}{2}.\frac{\pi}{2}\right) - \frac{1}{4}(sin\pi)\right]$$

$$= -4\left[\left(\frac{1}{2}.\frac{\pi}{2}\right) - 0\right] = -4\left[\frac{\pi}{4}\right] = -\pi$$

…..(1)

$$Curl\,\bar{F} = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix} = -(\imath + \jmath + \bar{k})$$

Unit normal vector $\bar{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\bar{\imath}+2y\bar{\jmath}+2z\bar{k}}{\sqrt{4x^2+4y^2+4z^2}} = x\bar{\imath} + y\bar{\jmath} + z\bar{k}$

Substituting the spherical polar coordinates, we get

$\bar{n} = sin\theta\cos\phi\,\bar{\imath} + sin\theta\sin\phi\,\bar{\jmath} + cos\theta\,\bar{k}$

$\therefore Curl\ \bar{F}.\bar{n} = -(sin\theta\cos\phi + sin\theta\sin\phi + cos\theta)$

$\iint Curl\ \ \bar{F}.\bar{n}\,ds = -\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}(sin\theta\cos\phi + sin\theta\sin\phi + cos\theta)sin\theta\ d\theta d\phi$

$$= -\int_0^{\pi/2} \lfloor sin\theta\sin\phi - sin\theta\cos\phi + \phi cos\theta \rfloor_0^{2\pi}\ sin\theta d\theta$$

$$= -2\pi\int_0^{\pi/2} cos\theta sin\theta d\theta = -\pi\int_0^{\pi/2} sin2\theta d\theta = (-\pi)\left[\frac{-cos2\theta}{2}\right]_0^{\pi/2}$$

$$=\frac{\pi}{2}(-1-1) = -\pi$$

…..(2)

From (1) and (2), we have

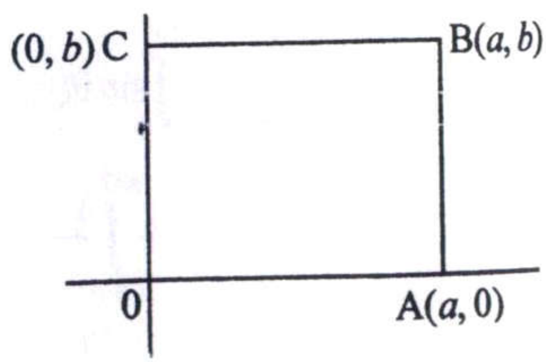
$\int_c \bar{F}.d\bar{r} = \int\int_s Curl\,\bar{F}.\bar{n}ds \;= -\pi$

$\therefore$ *Stoke*'s theorem is verified.

Example 16: Verify Stoke's theorem for $\bar{F} = (x^2+y^2)\bar{\imath} + 2xy\bar{\jmath}$ over the box bounded by the planes x=0,x=a,y=0,y=b.

[JNTU 2008 (Set No.1)]

Solution :



Stoke"s theorem states that $\int_c \bar{F}.d\bar{r} = \int_c Curl\,\bar{F}.\bar{n}d\,s$

Given $\bar{F} = (x^2-y^2)\bar{\imath} + 2xy\bar{\jmath}$

Curl$\bar{F}$= $\begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-y^2 & 2xy & 0 \end{vmatrix} = -\bar{\imath}(0-0) - \bar{\jmath}(0-0) + \bar{k}(2y+2y) = 4y\bar{k}$

R.H.S= $\int_c Curl\,\bar{F}.\bar{n}d\,s = \int 4y(\bar{k}.\bar{n})\,ds$

Let R be the region bounded by the rectangle

$(\bar{k}.\bar{n})ds = dx\ dy$

$\int_C Curl\ \bar{F}.\bar{n}d\ s = \int_{x=0}^{a}\int_{y=0}^{b} 4y\ dx\ dy = \int_{x=}^{a}\left[4\frac{y^2}{2}\right]_0^b dx = 2b^2\int_{x=0}^{a} 1dx$

$=2b^2(x)_0^a = 2ab^2$

To Calculate L.H.S

$\bar{F}.d\bar{r} = (x^2 - y^2)dx + 2xy\ dy$

Let) O=$(0,0), A = (a, 0), B = (a, b)$ and

C=(),b) are the vertices of the rectangle.

(i)Along the line OA

Y=0; dy=0, x ranges from 0 to a.

$\int_{OA} \bar{F}.d\bar{r} = \int_{x=0}^{a} x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$

(ii)Along the line AB

X=a; dx=0, y ranges from 0 to b.

$\int_{AB} \bar{F}.d\bar{r} = \int_{y=0}^{b}(2xy)\ dy = \left[2a\frac{y^2}{2}\right]_0^b = ab^2$

(iii)Along the line BC

Y=b; dy=0, x ranges from a to 0

$\int_{BC} \bar{F}.d\bar{r} = \int_{x=0}^{0}(x^2 - y^2)\ dx = \left[\frac{x^3}{3} - b^2x\right]_a^0 = 0 - \left(\frac{a^3}{3} - b^2a\right)$

$= ab^2 - \frac{a^3}{3}$

(iv) Along the line CO

X=,dx=0,y changes from b to 0

$\int_c \bar{F}\,.\,d\bar{r} = \int_{y=b}^{0} 2xy\,dy = 0$

Adding these four values

$\int_{co} \bar{F}\,.\,d\bar{r} = \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3} = 2ab^2$

L.H.S = R.H.S

Hence the verification of the stoke's theorem.

EXAMPLE 17: Verify Stoke's theorem for $\bar{F} = (x^2 - y^2)\bar{i} + 2xy\bar{j}$ over the box bounded by the planes x=0,x=a,y=0,y=b,z=c

[JNTU (K) June 2009 (Set No.1)]

Solution: Given $\bar{F} = (x^2 - y^2)\bar{i} + 2xy\bar{j}$

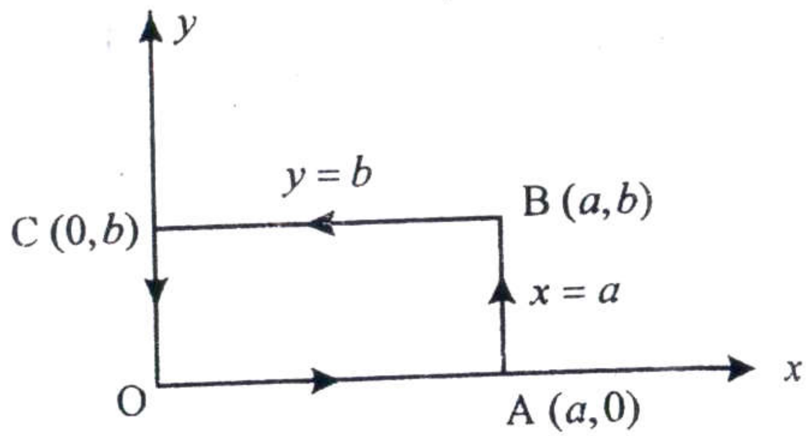
Let C denote the boundary of the box .

Then $\int_C \bar{F}\,.\,d\bar{r} = \int_c (x^2 - y^2)dx + 2xy\,dy$

The curve C is made up of four lines

OA,AB,BC and CO.

Along OA : y=0,dy=0



$\therefore \int_{OA}(x^2 - y^2)dx + 2xydy \;=\int_0^a x^2\,dx \;=\; \left(\frac{y^3}{3}\right)_0^a \;=\frac{a^3}{3}$ --------
-----------(1)

Along AB : x=a,dx=0

$\therefore \int_{AB}(x^2 - y^2)dx + 2xydy$ =2a $\int_0^b ydy$ = 2a $\left(\frac{y^2}{2}\right)_0^b \;=\; ab^2$ ----------
---------(2)

Along BC : y=b,dy=0

$\therefore \int_{AB}(x^2 - y^2)dx + 2xydy \;=\int_a^0 (x^2 - b^2)dx \;=\; \left(\frac{x^3}{3} - b^2x\right)_a^0$ =0 $-\left(\frac{a^3}{3} - ab^2\right)$

$=ab^2 - \frac{a^3}{3}$ --(3)

Along CO : x=0,dx=0

$\therefore \int_{CO}(x^2 - y^2)dx + 2xydy \;=\int_b^0 0.dy \;=\; 0$ ------------------
(4)

(1) + (2) + (3) + (4) gives $\int_c \bar{F}.d\bar{r} = \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3} = 2ab^2$---------------------------(5)

Again , Curl $\bar{F} = \nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$ =4y $\bar{k}$

Hence $\bar{n} = \bar{k}$.

$\int_s (\nabla \times \bar{F}).\bar{n}ds = \int_s 4ydxdy \quad = 4\int_{x=0}^{a}\int_{y=0}^{b} ydydx$

$=4 \int_0^a \left(\frac{y^2}{2}\right)_0^b dx = 2b^2 \int_0^a dx = 2b^2(x)_0^a = 2ab^2$ ----------------

(6)

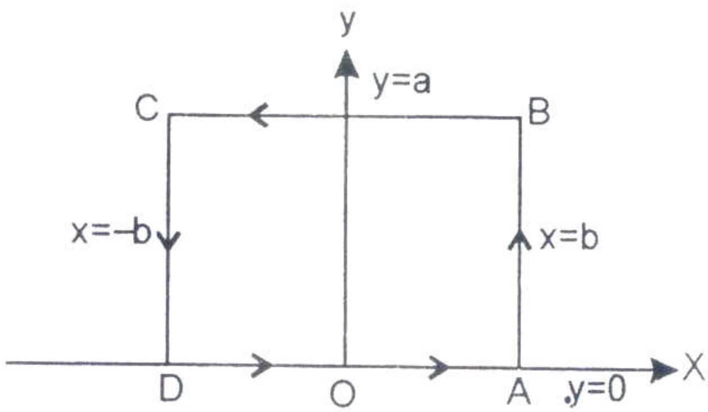
From (5) and (6) , we find that $\int_c \bar{F}.d\bar{r} = \iint_s curl\ \bar{F}.\bar{n}ds$

Hence the verification of the Stoke's theorem .

EXAMPLE 18: Verify Stoke's theorem for $\bar{F} = y^2\bar{i} - 2xy\bar{j}$ taken round the rectangle bounded by

x=$\pm b$, y=0,y=a.

Solution:



Curl $\bar{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & -2xy & 0 \end{vmatrix} = -4y\bar{k}$

For the given surface S, $\bar{n} = \bar{k}$

$(Curl\ \bar{F}).\bar{n} = -4y$

Now $\iint_s (Curl\ \bar{F}).\bar{n}\,dS = \iint_s -4y\,dx\,dy$

$$= \int_{y=0}^{a}\left[\int_{x=-b}^{b} -4y\,dx\right]dy$$

$$= \int_0^a -4xy\Big]_{-b}^{b} dy$$

$$= \int_0^a -8by\,dy = -4by^2\Big]_0^a = -4a^2b \quad \text{--------(1)}$$

$\int_c \bar{F}.d\bar{r} = \int_{DA} + \int_{AB} + \int_{BC} + \int_{CD}$

$\int \bar{F}.d\bar{r} = y^2\,dx - 2xy\,dy$

Along DA , y=0,dy=0, $\int_{DA} \bar{F}.d\bar{r} = 0$ ($\because\ \bar{F}\,d\bar{r} = 0$)

Along AB, x=b,dx=0

$$\int_{AB} \bar{F}\,.\,d\bar{r} = \int_{y=0}^{a} -2by\,dy = -by^2\Big|_0^a = -a^2 b$$

Along BC,y=a,dy=0

$$\int_{BC} \bar{F}\,.\,d\bar{r} = \int_{b}^{-b} a^2\,dx = -2a^2 b$$

Along CD, x=-b,dx=0

$$\int_{CD} \bar{F}\,.\,d\bar{r} = \int_{a}^{0} 2by\,dy = -by^2\Big|_a^0 = -a^2 b.$$

$$\int_{C} \bar{F}\,.\,d\bar{r} = 0 - a^2b - 2a^2b - a^2b = -4a^2 b \quad \text{-------(2)}$$

From (1),(2) $\int_C \bar{F}\,.\,d\bar{r} = \iint_S (Curl\,\bar{F}).\bar{n}\,dS$

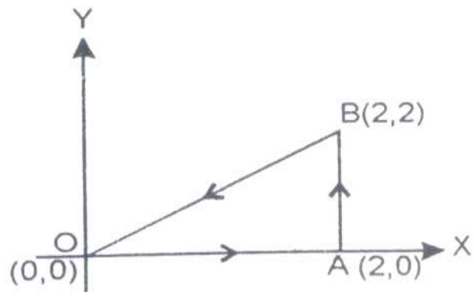
Hence the theorem is verified.

EXAMPLE 19: Using Stroke's theorem evaluate the integral $\int_C \bar{F}\,.\,d\bar{r}$ where

$\bar{F}=2y^2\,\bar{i}+3x^2\,\bar{j}$ -(2x+z)$\bar{k}$ and C is the boundary of the triangle whose vertices are (0,0,0),(2,0,0),(2,2,0).

<u>Solution:</u>

$$\text{Curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y^2 & 3x^2 & -2x - z \end{vmatrix} = 2\bar{j} + (6x-4y)\,\bar{k}$$



Since the z-coordinate of each vertice of the triangle is zero , the triangle lies in the xy-plane .

$\bar{n}$ =k

∴ (Curl $\bar{F}$).$\bar{n}$= 6x-4y

Consider the triangle in xy-plane .

Equation of the straight line OB is y=x.

By Stroke's theorem

$\int_s \bar{F}.d\bar{r} = \iint_s (Curl\,\bar{F}).\bar{n}dS$

$$=\int_{x=0}^{2}\int_{y=0}^{y=x}(6x-4y)\,dxdy \qquad = \int_{x=0}^{2}\left[\int_{y=0}^{x}(6x-4y)dy\right]dx$$

$$= \int_{x=0}^{2} 6xy-2y^2\Big|_0^x dx \quad = \int_0^2(6x^2-2x^2)dx$$

$$= 4\frac{x^3}{3}\Big|_0^2 \quad = \frac{32}{3}$$

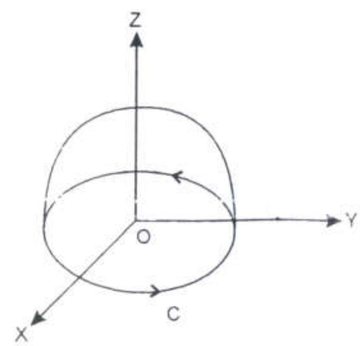
EXAMPLE 20: Evaluate $\iint_s(\nabla\times\bar{F}).\bar{n}dS$, where $\bar{F}$=(x^2+y-4)$\bar{i}$ + 3xy$\bar{j}$ +(2xz+z^2)$\bar{k}$ and S is the surface of, (i). The hemisphere $x^2+y^2+z^2$ =16 above the xy-plane , (ii) The paraboloid Z=4 - (x^2+y^2) above the xy-plane .

Solution: (i) Given $\bar{F}=(x^2+y-4)\bar{i} + 3xy\bar{j} +(2xz+z^2)\bar{k}$

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Then $d\bar{r}=\bar{i}dx + \bar{j}dy + \bar{k}dz$.

$\iint_s (\nabla \times \bar{F}).\bar{n}\,dS = \oint_c \bar{F}.d\bar{r}$



we have to find

$\oint_c (x^2 + y - 4)\,dx + 3xydy + (2xz + z^2)dz$

OBJECTIVE TYPE QUESTIONS

(1) For any closed surface S, $\iint_s curl\,\bar{F}\,\bar{n}\,dS$ =

(a) 0 (b) $2\bar{F}$ (c) $\bar{n}$ (d) $\oint \bar{F}\,d\bar{r}$

(2) if S is any closed surface enclosing a volume V and $\bar{F} = x\bar{i} + 2y\,\bar{j} + 3z\bar{k}$ then $\iint_s \bar{F}\,\bar{n}\,dS =$

(a) V (b)3V (c)6V (d)None

(3) If $\bar{r} = x\bar{i} + y\,\bar{j} + z\bar{k}$ then $\oint \bar{r}\,d\bar{r} =$

(a) 0 (b) $\bar{r}$ (c) x (d) None

(4) $\int \bar{r} \times \bar{n}\,dS$ =

(a) 0 (b) r (c) 1 (d) None

(5) $\int_s \bar{r} \,.\, \bar{n}\, dS =$

(a) V (b) 3V (c) 4V (d) None

(6) If $\bar{n}$ is the unit outward drawn normal to any closed surface then $\int_s \text{div}\, \bar{n}\, dV =$

(a) S (b)2S (c) 3S (d) None

(7) $\oint f\ \nabla f \,.\, d\bar{r} =$

(a) f (b)2f (c) 0 (d) None

(8) The value of the line integral $\int grad(x + y - z)\, d\bar{r}$ from (0, 1,-1) to (1 , 2, 0) is

(a) -1 (b) 0 (c) 2 (d) 3

(9) A necessary and sufficient condition that the line integral $\int_c A . dr$ =0 for every closed curve c is that

(a) div A=0 (b)div A 0 (c) curl A=0 (d) curl A 0

(10) If $\bar{F}$ =axi + byj+ czk where a, b, c are constants then $\iint \bar{F} . \bar{n} dS$ where S is the surface of the unit sphere is

(a) 0 (b) $\frac{4}{3}\pi(a + b + c)$ (c) $\frac{4}{3}\pi(a + b + c)^2$ (d) none

(11) $\int_V D \times \bar{F}\, dv =$____________________

(a) $\int_S \bar{n} \times \bar{F}\, ds$ (b) 0 (c) V (d) S

(12) $\int_V \phi \times dv =$ ____________

(a) $\int \bar{n}\ \phi ds$ (b) 0 (c) V (d) ϕ

(13) $\int f o g . d\bar{r}$= _______________

(a) 0 (b) $\int_S (\nabla f \times \bar{F} D g)$ (c) $\bar{r}$ (d) S

(14) $\iint_S x dydx + ydzdx + zdxdy$ where S: $x^2 + y^2 + z^2 = a^2$ as

(a) 4p (b) $\frac{4}{3}\pi a^3$ (c) $4\pi a^3$ (d) 4π

ANSWERS

(1) d (2) c (3) a (4) a (5) b (6) a (7) c

(8) d (9) c

(10) b (11) a (12) a (13) b (14) c

Assignment Mid-1

Subject:M2

1. Find a) $L(\int_0^t te^{-t}\sin 4t)$ b) $\int_0^\infty \frac{\cos at-\cos bt}{t}\,dt$ c) $L^{-1}(\frac{s^2}{s^4+4a^4})$ **(CO-1)**

2. Find a) $L(|\sin t|)$ b) Find L(f(t)) if $f(t)=\begin{cases}1 \text{ if } 0<t<2\\ 2 \text{ if } 2<t<4\\ 0 \text{ if } t>4\end{cases}$

   c) Find L(f(t)) if $f(t)=\begin{cases}\cos\left(t-\frac{\pi}{3}\right) & \text{if } t<\frac{\pi}{3}\\ 0 & \text{if } t>\frac{\pi}{3}\end{cases}$ **(CO-1)**

3. Solve the following differential equations by using laplace transforms
   a) $(D^2+n^2)x=a\sin(nt+\alpha)$ given x=Dx=0 at t=0.
   b) $y(t)=1-e^{-t}+\int_0^t y(t-u)\sin u\,du$ **(CO-2)**

4. Find a) $\int_0^2 x(8-x^3)^{\frac{1}{3}}dx$ b) $\int_0^{\frac{\pi}{2}} \cos^{\frac{3}{2}}\theta\,\sin^{\frac{5}{2}}\theta\,d\theta$ c) $\int_0^1 \frac{dx}{\sqrt{-\log x}}$ d) $\int_0^\infty e^{-x^{\frac{1}{n}}}dx$
   e) $\int_0^\infty 3^{-4x^2}dx$ **(CO-3)**

5. Prove that a) $B(m+\frac{1}{2},m+\frac{1}{2})=\frac{\pi}{m\,B(m,m)2^{4m-1}}$
   b) prove that $\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\Gamma\left(\frac{4}{n}\right)\ldots\ldots\Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{\frac{n-1}{2}}}{n^{\frac{1}{2}}}$ **(CO-3)**

Assignment Mid -2

1. a) Prove that $\nabla\times(\nabla\times\bar{a})=\nabla(\nabla.\bar{a})-\nabla^2\bar{a}$.
   b) Prove that $div(grad\,r^m)=m(m+1)r^{m-2}$ **OR** $\nabla^2(r^m)=m(m+1)r^{m-2}$. **(CO5)**
2. a) Prove that $curl(\bar{a}X\bar{b})=\bar{a}\,div\bar{b}-\bar{b}\,div\bar{a}+(\bar{b}.\nabla)\bar{a}-(\bar{a}.\nabla)\bar{b}$
   b) Find a and b such that the surfaces $ax^2-byz=(a+z)$ and $4ax^2y+z^3=4$.

   $cut\ orthogonally\ at\ (1,-1,2)$. **(CO5)**

3. a) Verify divergence theorem for $2x^2yi-y^2j+4xz^2k$ taken over the region of first

   Octant of the cylinder $y^2+z^2=9$, and x=0, x=2.

   b) Using Divergence theorem, evaluate $\iint_S (x\,dydz+y\,dzdx+z\,dxdy)$, where

   S: $x^2+y^2+z^2=a^2$. (CO6)

4. a) Verify Green's theorem for $\int_c$ (y-sinx)dx + cos xdy where C is the triangle formed by

the Points (0,0),(π/2,0),(π/2,1).

b) Verify Stokes theorem for $\overline{F}$=(x^2+y^2)i-2xyj taken round the rectangle bounded by the Lines x=$\pm a$,y=0,y=b. (CO6)

5.a) Find the volume of the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$ *and* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

b) Find the centroid of the area enclosed by parabola $y^2 = 4ax$ and the x-axis and latus rectum. (CO4)

Mid exam question paper along with sample Answers Scripts

MID-1

Answer any two questions 5x2=10

1a). Find L $(e^{-3t} \int_0^t \frac{sint}{t} dt)$

b). FindL^{-1}(log $(\frac{s+3}{s+4})$) (CO1)

2a). Solve $y^{11} = t \cos 2t$ given y (0) = $y^{11}(0) = 0$

b) y (t) = 1-e^{-t} +$\int_0^t y(t-u)\, sinu\, du$ (CO2)

3a) Prove that a)$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta\, cos^{2n-1}\theta\, d\theta$ and

b) Prove that $\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\Gamma\left(\frac{4}{n}\right) \dots\dots \Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{\frac{1}{2}}}$ (CO3)

4a) find a) $\int_0^1 \frac{dx}{\sqrt{-logx}}$

b) $\int_0^\infty 3^{-4x^2}$dx

c) Prove that $\int_0^1 (1-x^n)^{\frac{1}{n}} dx = \frac{1}{n} \frac{(\Gamma\left(\frac{1}{n}\right))^2}{2\Gamma\left(\frac{2}{n}\right)}$ (CO3)

MID-2

Answer any two questions 5x2=10

1a. Find the center of gravity of the area of the cardioids r=a(1+cosθ).
Using triple integral find the volume of the sphere whose radius is a units.
b) Find the area of the circle $x^2 + y^2$=a^2using double integration. (CO4)

2a) If $\vec{f}$ = $3x^2z^2y\bar{i}$+ $\bar{j}x^2z^2$+$2x^3yz\bar{k}$. Show that $\int_c f\,.d\bar{r}$ is independent of the path of integration. Hence evaluate the integral when C is any path joining (0,0,0) to (2,1,3).
b) Find the values of a and b so that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ intersect orthogonally at point (1,-1,2). (CO5)
3) Verify Green's theorem for $\oint[(3x^2 - 8y^2)dx + (4y - 6xy)dy]$ where c is the region bounded by x=0,y=0 and x+y=1. (CO6)
4a) Using Divergence theorem, evaluate $\iint_S (x\, dydz + y\, dzdx + z\, dxdy)$, where

S: $x^2 + y^2 + z^2 = a^2$.

b) If F= yi+(x-2xz)j-xyk, evaluate $\int(\nabla XF).nds$ where S is the surface of sphere $x^2 + y^2 + z^2 = a^2$ in xy plane. (CO6)

Scheme of Evaluation

Mapping of co's with po's

	Relationship Course Outcomes (CO) Programs Outcomes (PO)											
Course Outcomes	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
C01	3	2	2	2	-	-	-	-	-	-	-	-
C02	3	2	2	2	-	-	-	-	-	-	-	-
C03	3	2	2	2	-	-	-	-	-	-	-	-
C04	3	2	2	2	-	-	-	-	-	-	-	-
C05	3	2	2	2	-	-	-	-	-	-	-	-
CO6	3	2	2	2	-	-	-	-	-	-	-	-

1-LOW 2-MEDIUM 3-HIGH

Attainment of cos & pos (Excel sheet)

Question Bank

Unit-1

1. $L^{-1}(\frac{5s-2}{s^2(s+2)(s+3)})$

2.using convolution find $L^{-1}\left(\frac{1}{s(s^2+1)(s^2+4)(s^2+16)}\right)$

3.Evaluate L{e^t (cos 2t+(1/2)sinh 2t)}(2005 sep)

4 .Find the Laplace transform of e^{-3t} (2cos5t-3sin5t).(sep 2007)

5. Find the Laplace transform of e^{-t} (3sin 2t-5cosh 2t).(sep 2003)

6. Find the Laplace transform of e^{-at} sinh bt.(2003 sep)

7. Using the therom on transformation of derivatives, find the Laplace transform ofe^{at}.(2000

8.Laplace transform of integral.(2003 sep)

9.Find L^{-1} {s/(s^2-a^2)}.(may 2006)

10.Find inverse laplace transform of (s^2+s-2)/s(s+3)(s-2).(2005 sep)

11.Find inverse Laplace transform of (s+2)/(s^2-2s+5).(2003 sep)

12.Find inverse Laplace transform of (3s-14)/(s^2-4s+8).(may 2003).

Unit-2

1.Relation between beta and gamma functions.

2.$\int_0^{\frac{\pi}{2}} sin^5\theta cos^{\frac{7}{2}}\theta d\theta$

3.$\int_0^1 \frac{x^n}{\sqrt{1-x^2}} dx$

4.$\int_0^1 \frac{dx}{\sqrt{1-x^n}} dx$

6.$\int_0^\infty \frac{x^2}{1+x^4} dx$

7.$\int_0^\infty x^m e^{-ax^n} dx$

8.$\int_0^\infty \frac{x^c}{c^x} dx$

9.$\int_0^{\frac{\pi}{2}} sin^m x cos^n x\, dx$

10.$\int_0^1 x^m (\log \quad x)^n dx$

Unit-3

1.Find $\iint (x+y)^2\, dxdy$ over the area bounded by the ellipse $x^2/a^2+y^2/b^2=1$.(sept 2006)

2.Evaluate $\iint (x^2+y^2) dxdy$ in positive quadrant for which x+y< 1.(may 2006)

3.Evaluate $\iint (x^2+y^2) dxdy$ ove the area bounded by the ellipse $x^2/a^2+y^2/b^2=1$.(.(Dec 2010)

4.Evaluate $\iint r^3\, drd\theta$ over the area included between the circles r=2 sinθ and r=4 sinθ.(Dec 2010)

5.Evaluate the triple integral $\iiint xy^2z$ dx dy dz taken through the positive octant of the sphere $x^2+y^2+z^2=a^2$. (Dec 2010)

6.Evaluate $\iiint z^2$ dx dy dz taken over the volume bounded by the surfaces $x^2+y^2=a^2$,$x^2+y^2=z$ and Z=0.(may 1999)

7.Evaluate $\iiint xyz$ dx dy dz where V is the domain bounded by the coordinate planes and the plane x+y+z=1(Dec 2000)

8.Evaluate $\iiint xyz$ dx dy dz,where the domain V is bounded by the plane x+y+z=a and the

Coordinate planes.(sep 2006)

9.Find the area of the loop of the curve r=a(1+cosθ).(sep 2007)

10.Find the volume common to the cylinder $x^2+y^2=a^2$ and $x^2+z^2=a^2$.(Dec 2000)

11.Find volume bounded by the cylinder $x^2+y^2=4$,y+z=4 and z=0.(sep 2000)

12.Find the volume of the solid generated by the revolution of the cardioid r=a(1-cosθ).(may 2006)

13.Find the volume of the region bounded by $z=x^2+a^2$,z=0,x=-a,x=a,y=-a,y=a.(sep 2008)

14.Find the volume of the solid generated by the revolution of the cardioid r=a(1-cosθ) about its axis.(may 2007)

15.Find by double integral ,the volume of the solid bounded by $z=0,x^2+y^2=1$and x+y+z=3.(may 2010)

.

Unit-4

1. Find the work done by the force $\bar{F}$= (2y+3)i+xzj+(yz-x)k When it moves a particle from the point (0,0,0) to (2,1,1) along the Curve $x=2t^2$, y=t and $z=t^3$ (dec-2010)
2. Use divergence theorem to evaluate $\iint_s (y^2z^2i + z^2x^2j + z^2y^2k).\bar{n}ds$ where S is the part of the unit sphere above xy-plane (dec-2010)
3. If F and G are two vectors, then prove that $div(\bar{F}X\bar{G}) = \bar{F}curl\bar{G} - \bar{G}.curl\bar{F}$ (dec-2010)
4. Evaluate $\oint_c xdy + ydx$ where c is the loop of the Folium of D'cartes $x = \frac{3at}{1+t^3}, y = \frac{3at^2}{1+t^3}$ (dec-2010)
5. verify stoke's theorem for $F=(2x-y)i-yz^2j-y^2zk$ over upper half surface of $x^2+y^2+z^2=1$ bounded by its projection on the xy-plane (dec-2010)

Unit-5

1.evaluate Green's theorem $\int_C (x^2 - coshy)dx + (y + sinx)dy$ where c is the rectangle with vertices (0,0),(∏ ,0),(∏,1),(0,1) (dec-2010)

2. Find the directional derivative of f(x,y,z)=zx^2-xyz at the point (1,3,1) in the direction of the vector 3i-2j+k (jun-2011)

3. Evaluate the line integral $\int_C (x^2 + xy)dx + (x^2 + y^2)dy$ where c is the square formed by the Lines y=±1,x=±1 (jun-2011)

4. In what direction from the point (-1,1,2) is the directional derivative of ø(x,y,z) = xy^2z^3 a maximum what is the magnitude of this maximum. (jun-2011)

5.find the circulation of F^- round the curve c where $\bar{F} = (e^x siny)i + (e^x cosy)j$ and c is the rectangle whose vertices are (0,0),(1,0),(1,∏/2),(0,∏/2) (jun-2011)

6. Prove that if ø and φ are scalar functions. Then prove that $\nabla \emptyset X \nabla \varphi$ is solenoidal (jun-2011)

Power point presentation

Websites/URLS/e-Resources

1.http://mathforcollege.com/nm/nbm/gen/05inp/
2.http://www.mece.panam.edu/~jakypuros/Teaching/MECE2450/Notes/PolynomialInterpolation.pdf
3.http://nm.mathforcollege.com/topics/fft_continuous.html
4.http://users.ece.gatech.edu/~mcclella/2025/labs-s01/Lab11s01.pdf
5.http://www.enm.bris.ac.uk/admin/courses/EMa2/PDEs/PDES_0203/EMa2_pdes_notes.pdf
6.http://ar-new.mak.ac.ug/academics/courses/partial-differential-equations.html-0
7.http://maths.york.ac.uk/www/Vector1-0910
8.http://www.youtube.com/watch?v=NG9hkGQwT3k
9.http://www.youtube.com/watch?v=sDn5cc-8gHY
10.http://www.youtube.com/watch?v=lCNHXhLg2dI
11. http://www.youtube.com/watch?v=oYsb4rW2GUU
12. http://www.youtube.com/watch?v=U8riFeiiu3s
13. http://www.youtube.com/watch?v=6ozQ9INV59s
14.http://www.cengage.com/aushed/instructor.do?product_isbn=9780534370145
15. http://na.uni-tuebingen.de/~lubich/pcam-ode.pdf
16. http://teacher.buet.ac.bd/cfc/CE205/CE205_Lec1.pdf
17. http://en.wikibooks.org/wiki/Numerical_Methods/Equation_Solving
18. http://www3.nd.edu/~powers/ame.60611/M.pdf